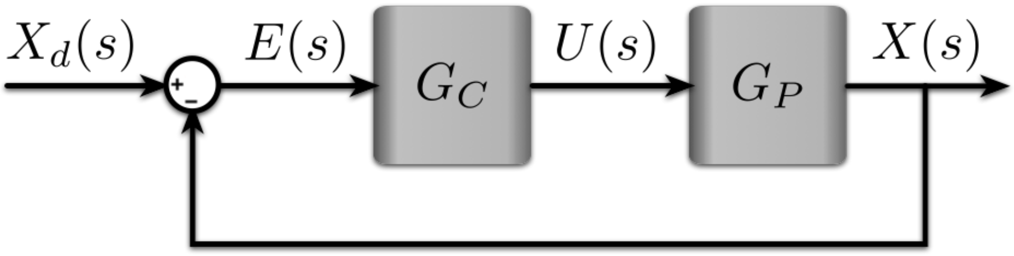
Block Diagram Models (cont.)

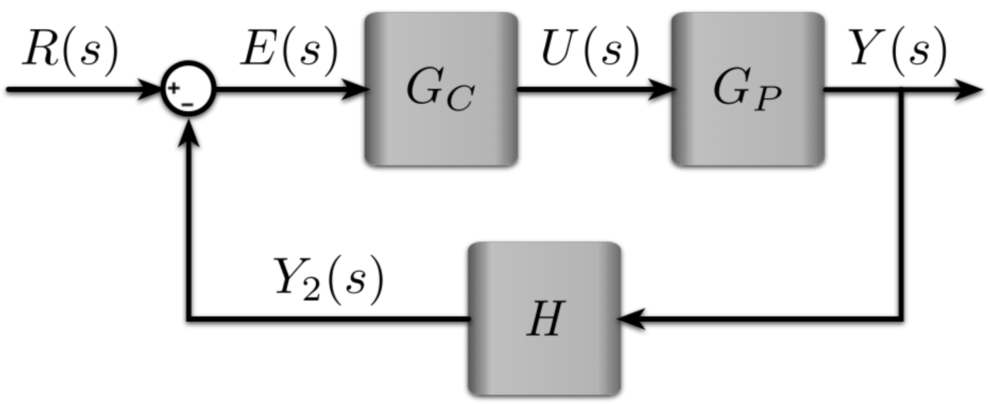


Q: Should we always feed back exactly what we measure (X in this case)?

No, particularly in practice.

Real sensors are noisy, have drift, etc. and you may not be able to measure the state you care about.

So, this is another common block diagram structure.



where $H(s)$ is the TF that may represent a signal processing operation and/or sensor properties

So, depending on the system

- Y_2 could be an estimate of Y
- Y_2 could be a filtered version of Y
- some combination of filtered and estimated states

Chapter 3 - State Variable Models

So far, we've looked at systems in the Laplace (or "s") domain

We can also model and analyze systems in the time domain.

Time domain representations allow us to look at:

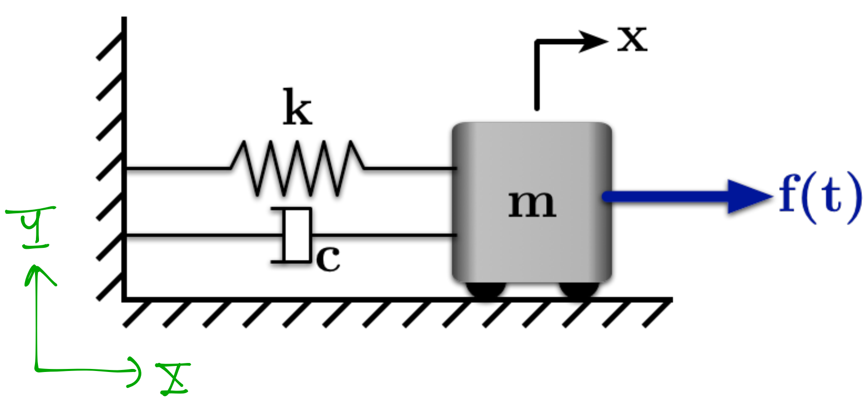
- Time varying systems ← system parameters change as a function of time.
- nonlinear systems
- and leverage linear algebra techniques for analysis

State Variables of a Dynamic System (Sec. 3.2)

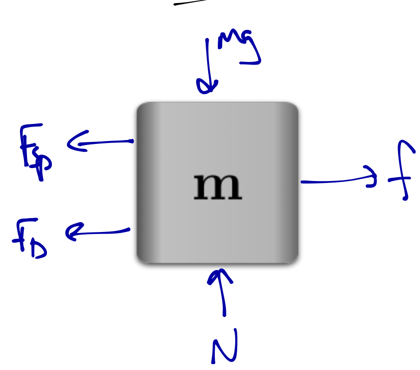
states are a set of variables that describe the system (excludes inputs)

usually represent displacements and rates of change of system degrees-of-freedom

Let's look at an example. (we'll take a slightly different path than the book, but arrive at a similar place.)



FBD



$m\ddot{x} - F_D - F_{sp} = f \longrightarrow m\ddot{x} + c\dot{x} + kx = f$

Define state vector $\bar{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$ ← Be careful here. Convention dictates that state vectors are written as x, even if that variable is being used to represent a system DOF.

Write $\dot{\bar{X}} = \underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix}}_{\text{write this in terms of system state variables}}$

In some of the Jupyter Notebooks, I'll use $\bar{w}$ for the state vector to avoid this.

State Variables of a Dynamic System (cont.)

Eq of motion – $m\ddot{x} + c\dot{x} + kx = f$

States $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$

Write $\dot{\bar{x}} = \begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix}$ ← just x_2 ; ← just the eq. of motion $\ddot{x} = \frac{-k}{m}x - \frac{c}{m}\dot{x} + \frac{f}{m} \rightarrow \ddot{x} = \frac{-k}{m}x_1 - \frac{c}{m}x_2 + \frac{f}{m}$

$$\dot{\bar{x}} = \begin{bmatrix} x_2 \\ \frac{-k}{m}x_1 - \frac{c}{m}x_2 + \frac{f}{m} \end{bmatrix}$$

This system of 1st order diff eq describes the dynamics of the system

The State Differential Equation (Sec. 3.3)

We want to write that equation in the form

$\dot{\bar{x}} = A\bar{x} + B\bar{u}$ where $\bar{x}$ = state vector and $\bar{u}$ is a vector of the inputs

$$\underbrace{\dot{\bar{x}}}_{\dot{\bar{x}}} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\bar{x}} + \underbrace{\begin{bmatrix} 0 \\ 1/m \end{bmatrix}}_{B} \underbrace{f}_{\bar{u}}$$

$$\dot{\bar{x}} = A \bar{x} + B \bar{u}$$

The output can also be written as a combination of states and the inputs

$\bar{y} \equiv$ vector of outputs

$y = C\bar{x} + D\bar{u}$

State-space Form

$\dot{\bar{x}} = A\bar{x} + Bu$

$\bar{y} = C\bar{x} + D\bar{u}$

The State Differential Equation (cont.)

This form generalizes to systems with more states and inputs

For system with n states and m inputs

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \bar{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$$

$$\dot{x}_1 = a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n + b_{11}u_1 + \dots + b_{1m}u_m$$

$$\dot{x}_2 = a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n + b_{21}u_1 + \dots + b_{2m}u_m$$

$$\vdots$$

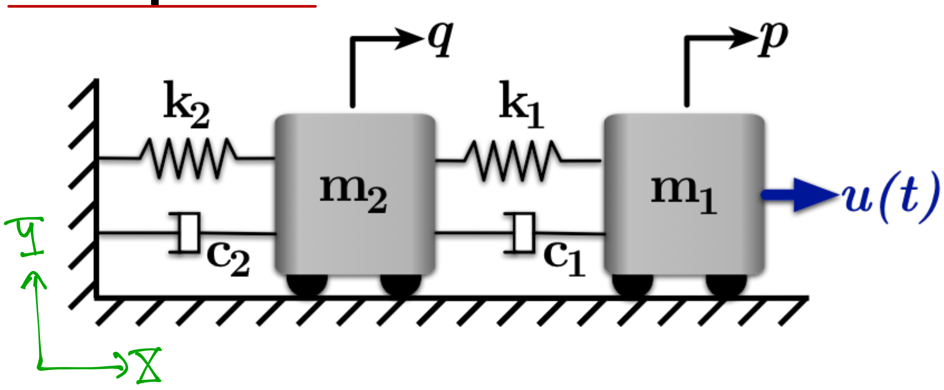
$$\dot{x}_n = a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n + b_{n1}u_1 + \dots + b_{nm}u_m$$

In matrix form

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix}}_{\dot{\bar{x}}} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{\bar{x}} + \underbrace{\begin{bmatrix} b_{11} & \dots & b_{1m} \\ b_{21} & \dots & b_{2m} \\ \vdots & & \\ b_{n1} & \dots & b_{nm} \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}}_{\bar{u}}$$

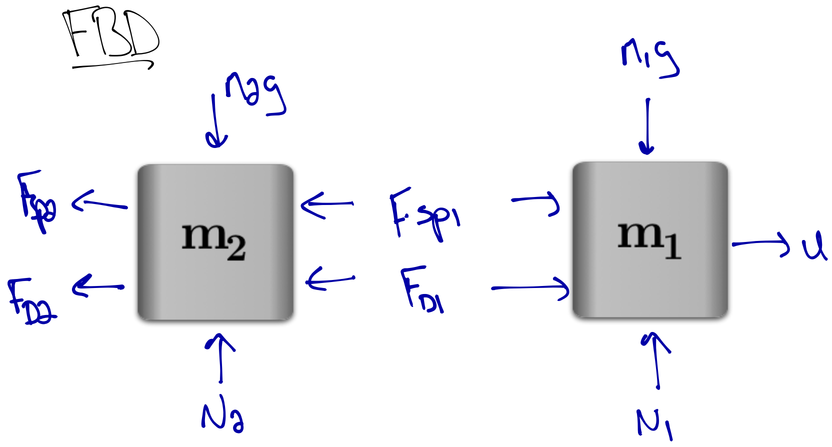
$$\underset{n\times1 \text{ vector}}{\dot{\bar{x}}} = \underset{n\times n \text{ matrix}}{A}\ \underset{n\times1 \text{ vector}}{\bar{x}} + \underset{n\times m \text{ matrix}}{B}\ \underset{m\times1 \text{ vector}}{\bar{u}}$$

Example 3.1



Write the state space form of the eq. of motion for this system.

FBD



$$F_{sp1} = k_1\delta_1 = k_1(q-p)$$
$$F_{D1} = c_1\dot{\delta}_1 = c_1(\dot{q}-\dot{p})$$
$$F_{sp2} = k_2\delta_2 = k_2 q$$
$$F_{D2} = c_2\dot{\delta}_2 = c_2\dot{q}$$

$$m_1\ddot{p} = k_1(q-p) + c_1(\dot{q}-\dot{p}) + u$$
$$m_2\ddot{q} = -k_1(q-p) - c_1(\dot{q}-\dot{p}) - k_2 q - c_2\dot{q} = k_1 p + c_1\dot{p} - (k_1+k_2)q - (c_1+c_2)q$$

Want equations in the form

$\dot{x}$ = linear combination of states and inputs

$$\ddot{p} = -\frac{k_1}{m_1}p - \frac{c_1}{m_1}\dot{p} + \frac{k_1}{m_1}q + \frac{c_1}{m_1}\dot{q} + \frac{u}{m}$$
$$\ddot{q} = \frac{k_1}{m_2}p + \frac{c_1}{m_2}\dot{p} - \frac{(k_1+k_2)}{m_2}q - \frac{(c_1+c_2)}{m_2}\dot{q}$$

Q: What should we choose as the state vector?

$$\bar{x} = \begin{bmatrix} p \\ \dot{p} \\ q \\ \dot{q} \end{bmatrix} \quad \text{or} \quad \bar{x} = \begin{bmatrix} p \\ q \\ \dot{p} \\ \dot{q} \end{bmatrix} \quad \text{or} \quad \bar{x} = \begin{bmatrix} p-q \\ q \\ \dot{p}-\dot{q} \\ \dot{q} \end{bmatrix} \quad \text{or} \quad \dots$$

Let's use this one. It's the most common ordering of states we'll use.

The book uses this form.

Example 3.1 (cont.)

$$\ddot{p} = -\frac{k_1}{m_1}p - \frac{c_1}{m_1}\dot{p} + \frac{k_1}{m_1}q + \frac{c_1}{m_1}\dot{q} + \frac{u}{m}$$

$$\ddot{q} = \frac{k_1}{m_2}p + \frac{c_1}{m_2}\dot{p} - \frac{(k_1+k_2)}{m_2}q - \frac{(c_1+c_2)}{m_2}\dot{q}$$

$$\bar{x} = \begin{bmatrix} p \\ \dot{p} \\ q \\ \dot{q} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \quad \text{so} \quad \dot{\bar{x}} = \begin{bmatrix} \dot{p} \\ \ddot{p} \\ \dot{q} \\ \ddot{q} \end{bmatrix} = \begin{bmatrix} \dot{p} \\ -\frac{k_1}{m_1}p - \frac{c_1}{m_1}\dot{p} + \frac{k_1}{m_1}q + \frac{c_1}{m_1}\dot{q} + \frac{u}{m} \\ \dot{q} \\ \frac{k_1}{m_2}p + \frac{c_1}{m_2}\dot{p} - \frac{(k_1+k_2)}{m_2}q - \frac{(c_1+c_2)}{m_2}\dot{q} \end{bmatrix}$$

Now write this as a function of the states

$$\dot{\bar{x}} = \begin{bmatrix} \dot{p} \\ \ddot{p} \\ \dot{q} \\ \ddot{q} \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{k_1}{m_1}x_1 - \frac{c_1}{m_1}x_2 + \frac{k_1}{m_1}x_3 + \frac{c_1}{m_1}x_4 + \frac{u}{m} \\ x_4 \\ \frac{k_1}{m_2}x_1 + \frac{c_1}{m_2}x_2 - \frac{(k_1+k_2)}{m_2}x_3 - \frac{(c_1+c_2)}{m_2}x_4 \end{bmatrix}$$

Now, write in matrix form $\dot{\bar{x}} = A\bar{x} + B\bar{u}$

$$\boxed{\dot{\bar{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1}{m_1} & -\frac{c_1}{m_1} & \frac{k_1}{m_1} & \frac{c_1}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{k_1}{m_1} & \frac{c_1}{m_2} & -\frac{(k_1+k_2)}{m_2} & -\frac{(c_1+c_2)}{m_2} \end{bmatrix}\bar{x} + \begin{bmatrix} 0 \\ 1/m \\ 0 \\ 0 \end{bmatrix} u}$$

Q: What is/are the output(s)?

p, q, or p and q

p as output − $y = p \rightarrow y = x_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}\bar{x}$

q as output − $y = q \rightarrow y = x_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}\bar{x}$

p and q as outputs − $y = \begin{bmatrix} p \\ q \end{bmatrix} \rightarrow y = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\bar{x}$

output can also be some linear combo of states and input. Just write in matrix form