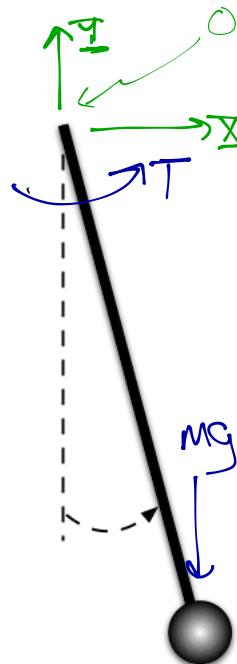
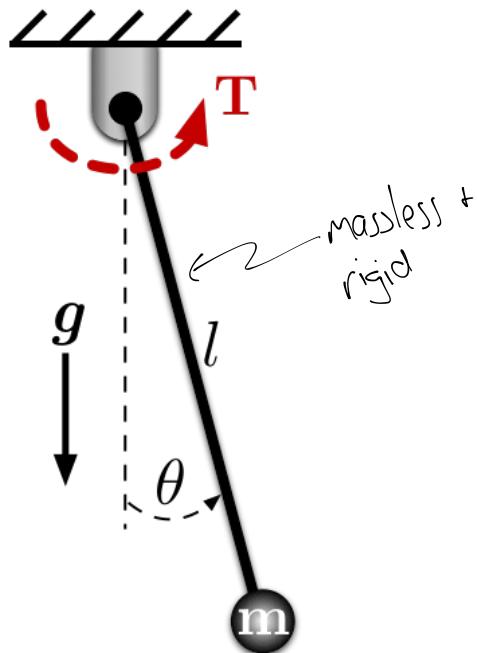


(Simple) Transfer Function Example



The system is in pure rotation about point O, so we can sum moments about that point to determine the eq. of motion

$$\sum \bar{M}_O = \dot{H}_O \quad \dot{H}_O = \text{rate of change of ang. vel.} \\ = \frac{d}{dt}(I_O \dot{\theta})$$

$$\sum \bar{M}_O = I_O \ddot{\theta} \bar{K} \quad I_O = \text{moment of inertia about point O}$$

$$\sum \bar{M}_O = T \bar{K} + \underbrace{(l \sin \theta \bar{I} - l \cos \theta \bar{J})}_{\text{moment created by force } \bar{r} \times \bar{F}} \times (-mg \bar{J}) \\ = T \bar{K} - mgl \sin \theta \bar{K}$$

$$I_O \ddot{\theta} \bar{K} = T \bar{K} - mgl \sin \theta \bar{K} \quad \leftarrow \text{all in the K direction, as they better be for this system}$$

$$I_O \ddot{\theta} + mgl \sin \theta = T$$

Q: What is I_O ? - point mass in pure rotation about a point at a distance r away is mr^2 , so here

$$I_O = ml^2$$

$$ml^2 \ddot{\theta} + mgl \sin \theta = T \quad \leftarrow \text{The eq. of motion for this system}$$

Q: Is it linear? - No

Q: So, can we write a TF?

- No. Need to linearize first.

(Simple) Transfer Function Example (cont.)

Let's assume that θ is "small", $\theta \approx 0$

$$\sin \theta \approx \theta \quad \cos \theta \approx 1$$

$$ml^2\ddot{\theta} + mgl\dot{\theta} = T \rightarrow \text{Laplace Transform} \rightarrow ml^2s^2\theta(s) + mgl\theta(s) = T(s)$$
$$(ml^2s^2 + mgl)\theta(s) = T(s)$$

$$\frac{\theta(s)}{T(s)} = \frac{1}{ml^2s^2 + mgl}$$

QV

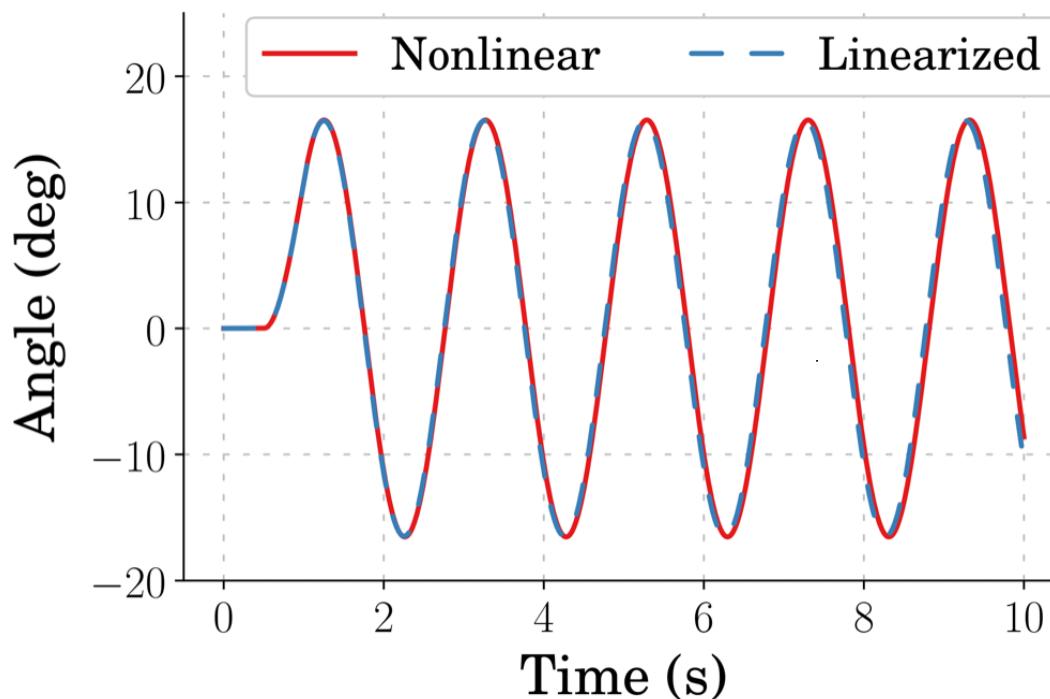
divide by ml^2

$$\ddot{\theta} + \frac{g}{l}\dot{\theta} = \frac{1}{ml^2}T$$

$$(s^2 + \omega_n^2)\theta(s) = \frac{1}{ml^2}T(s), \text{ where } \omega_n^2 = \sqrt{\frac{g}{l}}$$

$$\rightarrow \frac{\theta(s)}{T(s)} = \frac{1/ml^2}{s^2 + \omega_n^2}$$

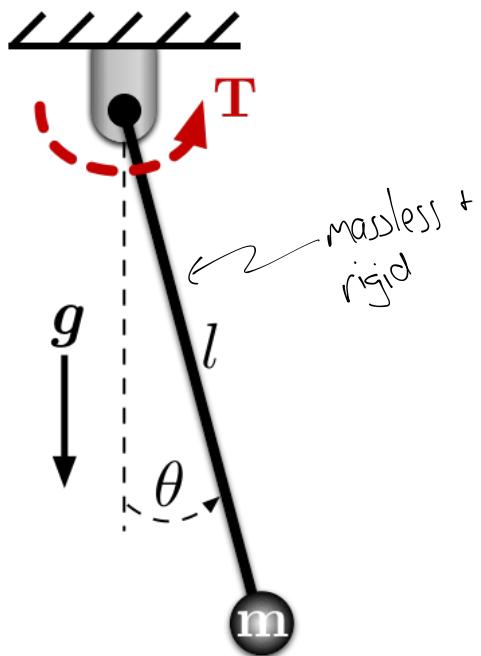
Q: How good is this linear approx? ... It depends.



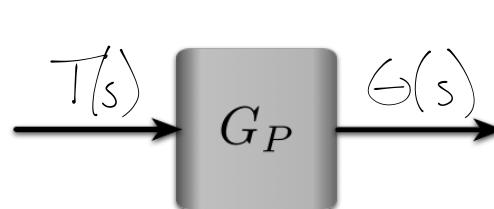
Block Diagram Models (Sec. 2.6)

Block diagrams represent operations on a signal with a system.

In our previous example, we had input T and output θ .



The block diagram representation of that system may look like:



Note: often see without the (s)

where $T(s)$ is the Laplace Transform of $T(t)$ and $\theta(s)$ is the Laplace Transform of $\theta(t)$

Blocks simply multiply their input by the block's content.

Q: What goes "inside" the block in this case? What is G_P ?

We want $\theta = G_P T$, we previously found a linearized version of this system and got its Transfer Function:

$$\frac{\theta(s)}{T(s)} = \frac{1/m l^2}{s^2 + \omega_n^2} \leftarrow \text{This is } G_P$$

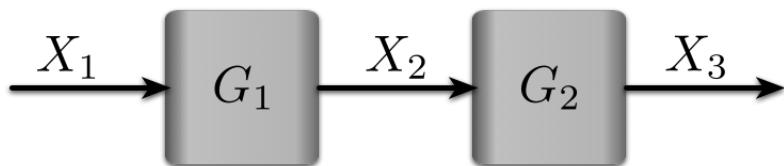
$$\theta = \left[\frac{1/m l^2}{s^2 + \omega_n^2} \right] T$$

G_P

Block Diagram Models (cont.)

Obviously, these block diagrams can get more complex. Their operation is always the same and we can use fairly basic math to simplify them.

Let's start simple by reducing



$$X_3 = G_2 X_2 \quad \text{and} \quad X_2 = G_1 X_1$$

substitute

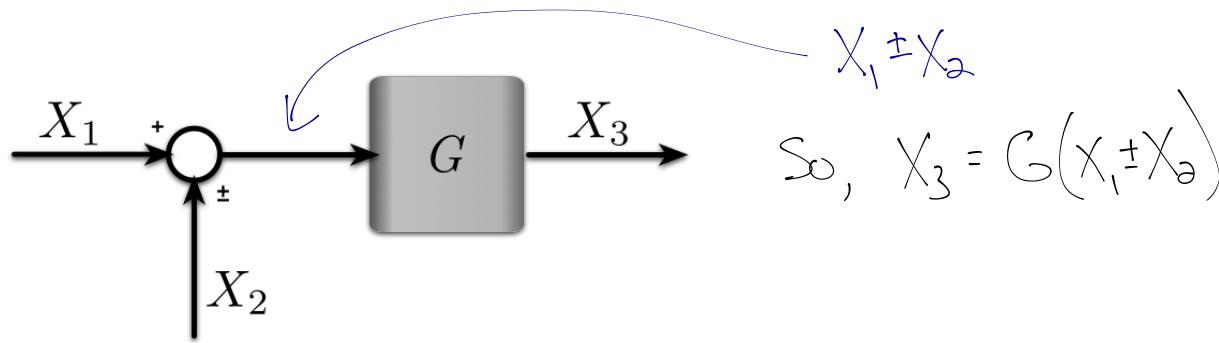
$$X_3 = G_2 G_1 X_1 \quad \text{or} \quad X_3 = G_1 G_2 X_1$$

So, we can redraw the block diagram as



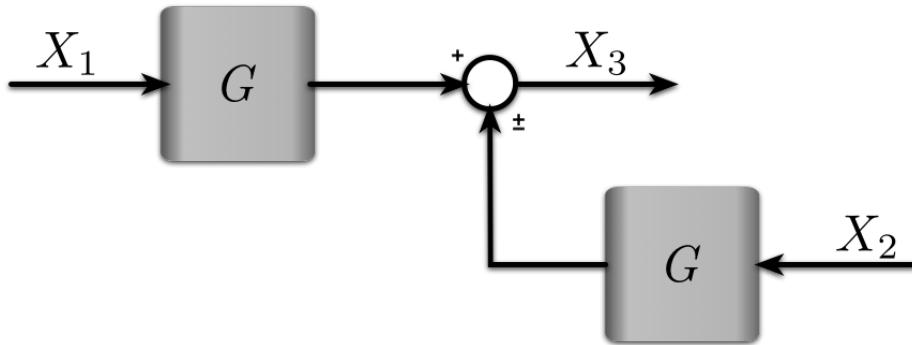
Block Diagram Models (cont.)

Signals can also add/subtract

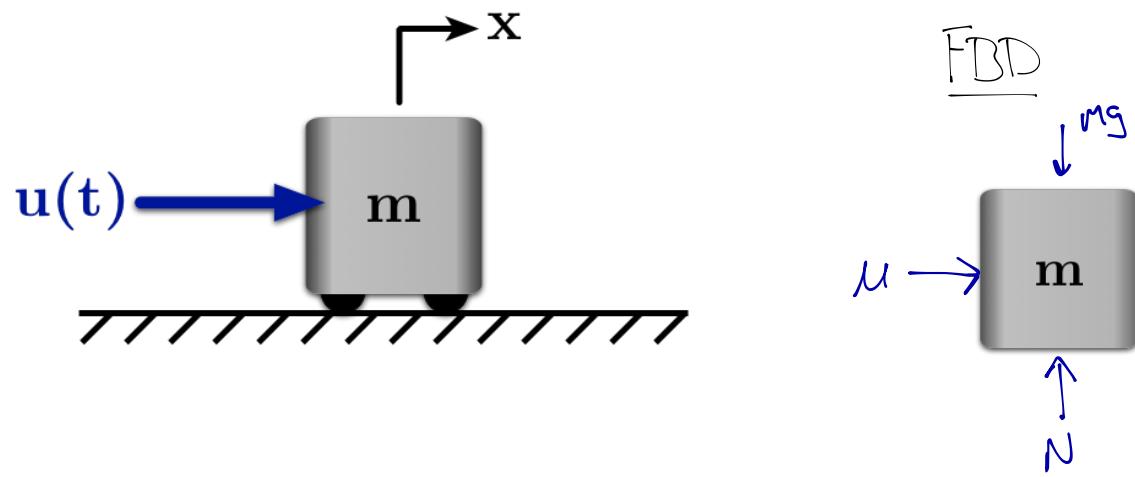


Q: Can we draw an equivalent diagram with a different structure?

$$X_3 = G(X_1 \pm X_2) = GX_1 \pm GX_2 \quad \leftarrow \text{"normal" algebra rules apply}$$



Block Diagram Example



Eq of motion:

$$m\ddot{x} = u$$

Q: What is the s-domain (after Laplace Transform) version of this eq.?

Remember that, assuming zero initial cond., $s = \frac{d}{dt}$, so

$$ms^2X(s) = U(s)$$

Q: So what is the transfer function for this system?

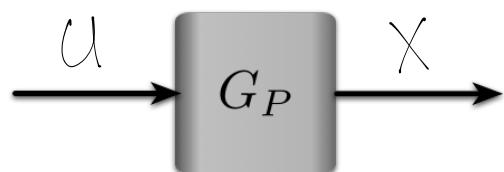
Output
Input

output is $X(s)$, the position of the mass

input is u , the force

$$\frac{X(s)}{U(s)} = \frac{1}{ms^2}$$

Q: How could we draw the block diagram for this system?

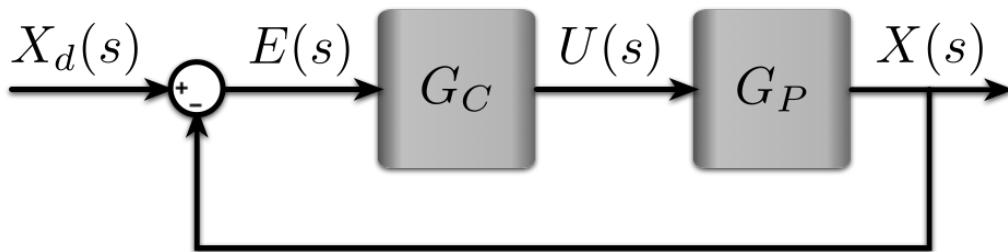


Block Diagram Example (cont.)

Now let's think back to the 1st day of class. Our objective was to move a mass to some desired position x_d .

We decided that if we could measure the mass's position + velocity we could pick the correct force input.

Q: What does that look like on a block diagram?



The output is fed back and used to generate a sys. command

This is a Feedback Controller

Let's look at this block diagram more closely.

Goal: A single block representing all the total system.

$$\text{Know } G_p = \frac{X}{U} = \frac{1}{m s^2}$$

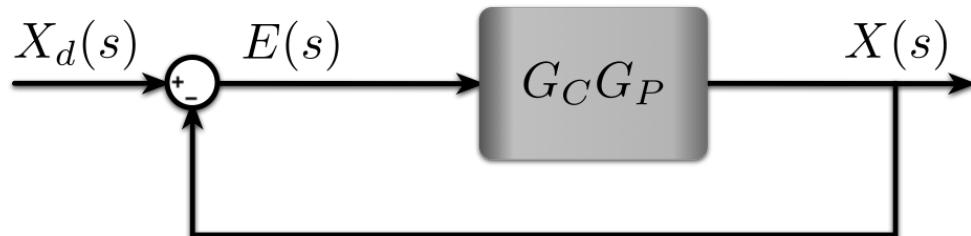
$$E = X_d - X$$

$G_c = ?$ ← Let's come back to this in a minute

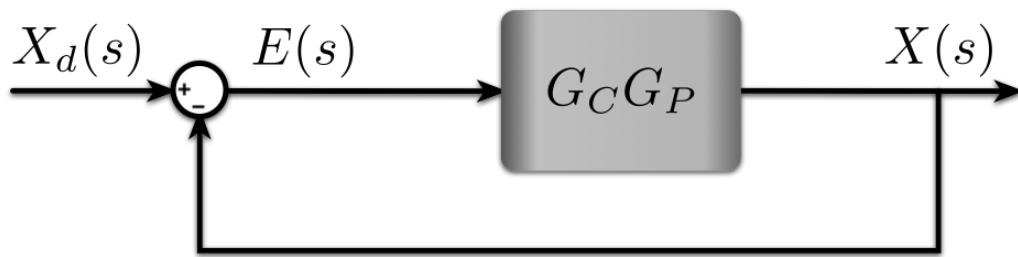
$U = G_c E$ ← G_c is an operator that takes the error from desired and generates a force, U , based on it

$$X = G_p U = G_p(G_c E)$$

$$X = G_c G_p E$$



Block Diagram Example (cont.)



$$X = G_C G_P E \quad E = X_d - X$$

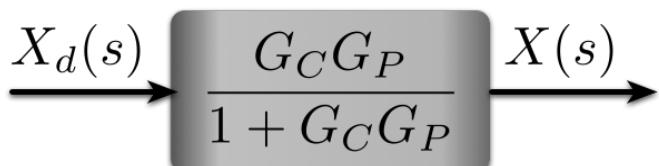
$$X = G_C G_P (X_d - X)$$

$$= G_C G_P X_d - G_C G_P X$$

$$X + G_C G_P X = G_C G_P X_d$$

Want output \rightarrow
$$\frac{X_d}{X} = \frac{G_C G_P}{1 + G_C G_P}$$

This is the closed-loop TF for this system. It includes the effects of the feedback loop.



Q: What are the poles of this closed-loop TF?

The roots of $1 + G_C G_P$ (where $1 + G_C G_P = 0$)

Q: What are the zeros of this TF?

The roots of $G_C G_P$ (where $G_C G_P = 0$)

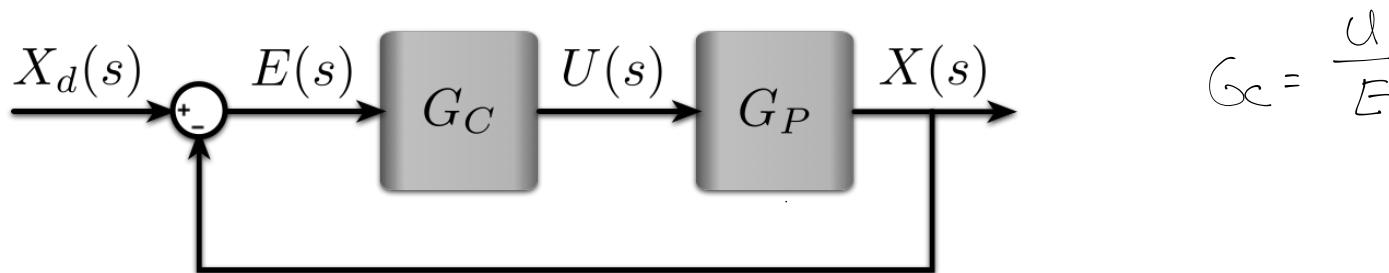
Block Diagram Example (cont.)

Q: What should G_C be?

On day 1, we require that $u(t) = k_p(\underline{x_d - x}) + k_d(\dot{\underline{x}}_d - \dot{x})$ might be a good choice.

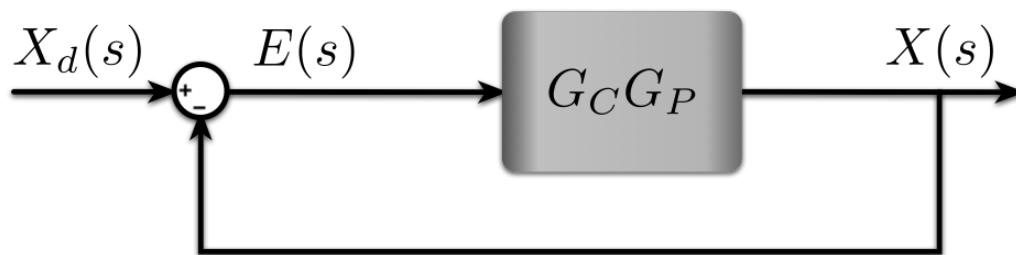
$$u = k_p e + k_d \dot{e}$$

$$u(s) = k_p E(s) + k_d (sE(s))$$



$$G_C = \frac{U}{E} = \frac{k_d s + k_p}{1}$$

already know $G_P = \frac{1}{m s^2}$



$$X(s) = G_C G_P E(s) \rightarrow \frac{X(s)}{E(s)} = G_C G_P = \frac{k_d s + k_p}{m s^2}$$

Q: What does this TF tell us?
How the output changes as a function of error

This is also called the open-loop transfer func. We'll see later that it is useful for control design.

$$\frac{X}{X_d} = \frac{G_C G_P}{1 + G_C G_P}$$

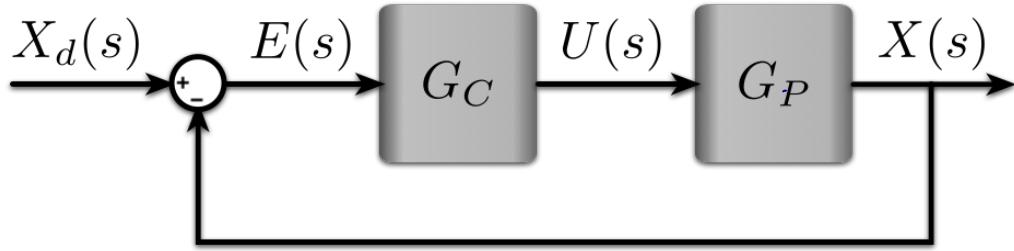
$$X = \left(\frac{G_C G_P}{1 + G_C G_P} \right) X_d$$

$$\frac{X}{X_d} = \frac{\left(\frac{k_d s + k_p}{m s^2} \right)}{1 + \left(\frac{k_d s + k_p}{m s^2} \right)} = \frac{k_d s + k_p}{m s^2 + k_d s + k_p}$$

} closed-loop TF for our proposed controller.

It tell us how the system respond for a given input X_d

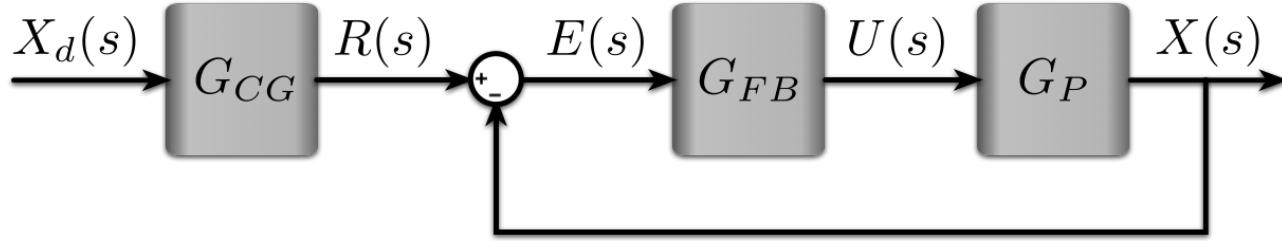
Block Diagram Models (cont.)



Q: Should our desired system state always be the input?

No. We can (or may need to) generate a more appropriate reference input, usually called R .

$r(t)$ is a function of $x_d(t)$, so we can add another block to our system to represent this.



$R = G_{CG} X_d \Rightarrow \frac{R}{X_d} = G_{CG} \leftarrow \text{command generator block.}$ You won't often see this in textbooks, but it is a powerful tool in our toolbox.