MCHE 474: Control Systems
Fall 2017 – Homework 3

Assigned:	Monday, September 25th
Due:	Friday, September 29th, 5pm

Assignment: From *Modern Control Systems (13th Edition)* by Richard Dorf and Robert Bishop, solve problems:
E4.2, E4.3, E4.4, E4.12, P4.2, E5.2, P5.3

Submission: Submission is not required. Solutions will be posted shortly after the due date listed above.

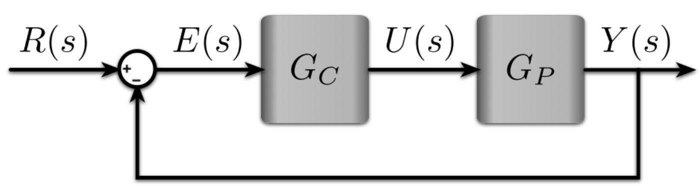
Problem E4.2

E4.2 A closed-loop system is used to track the sun to obtain maximum power from a photovoltaic array. The tracking system may be represented by a unity feedback control system and

$$G_c(s)G(s) = \frac{100}{\tau s + 1},$$

where $\tau = 3$ s nominally. (a) Calculate the sensitivity of this system for a small change in τ. (b) Calculate the time constant of the closed-loop system response.

The block diagram for this system is:



So, the closed-loop TF is

$$\frac{Y}{R} = \frac{G_cG_p}{1+G_cG_p} = T(s)$$

a)
$$S_G^T = \frac{\partial T/T}{\partial G/G} = \frac{\partial T}{\partial G}\cdot\frac{G}{T} = \left[\frac{G_c}{1+G_cG_p} + \frac{-G_c^2G_p}{(1+G_cG_p)^2}\right]\left[\frac{G_p(1+G_cG_p)}{G_cG_p}\right]$$

$$= \left[\frac{\cancel{G_c}(1+G_cG_p) - G_c^{\cancel{2}}G_p}{(1+G_cG_p)^{\cancel{2}}}\right]\left[\frac{\cancel{G_p}\cancel{(1+G_cG_p)}}{\cancel{G_c}\cancel{G_p}}\right]$$

$$= \frac{1}{1+G_cG_p}$$

← As was shown in Eq 4.13 in the book

$$S_G^T = \frac{1}{1+\frac{100}{\tau s+1}} = \frac{\tau s+1}{\tau s+101}$$

when $\boxed{\tau = 3s \quad S_G^T = \frac{3s+1}{3s+101}}$

b)
$$\frac{Y}{R} = \frac{G_cG_p}{1+G_cG_p} = \frac{\frac{100}{\tau s+1}}{1+\left(\frac{100}{\tau s+1}\right)} = \frac{100}{\tau s+101} = \frac{100/101}{\underbrace{\frac{\tau}{101}}s+1}$$

This is the time constant of the closed-loop system so

$$\boxed{\tau_c = \frac{\tau}{101} = \frac{3}{101} = 0.029s}$$

Problem E4.3

E4.3 A robotic arm and camera could be used to pick fruit, as shown in Figure E4.3(a). The camera is used to close the feedback loop to a microcomputer, which controls the arm [8, 9]. The transfer function for the process is

$$G(s) = \frac{K}{(s+10)^2}.$$

(a) Calculate the expected steady-state error of the gripper for a step command A as a function of K. (b) Name a possible disturbance signal for this system.

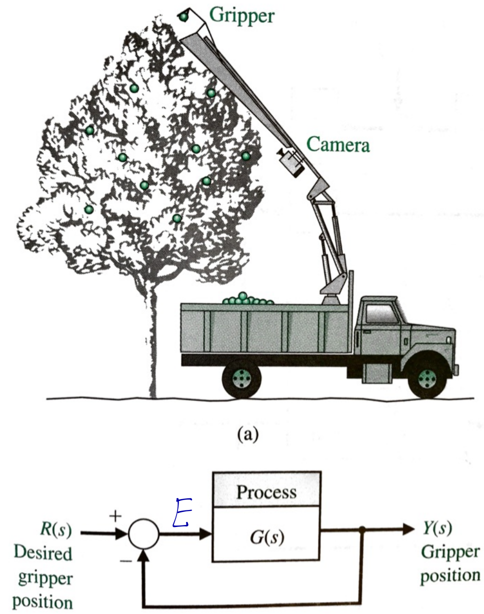


FIGURE E4.3 Robot fruit picker.

a) We first need to find the TF representing the tracking error of the closed-loop system:

$$E = R - Y = R - GE$$

$$(1+G)E = R \rightarrow E = \left[\frac{1}{1+G}\right]R$$

$$E = \left[\frac{(s+10)^2}{(s+10)^2 + K}\right]R$$

$$E = \frac{s^2 + 20s + 100}{s^2 + 20s + (100+K)}\, R$$

A step command of A in $r(t) \rightarrow R(s) = \frac{A}{s}$

$$E = \left[\frac{s^2 + 20s + 100}{s^2 + 20s + (100+K)}\right]\left[\frac{A}{s}\right]$$

To find steady-state error, use the final value theorem:

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$$

$$\boxed{\lim_{s\to 0} sE = \lim_{s\to 0} \frac{A(s^2 + 20s + 100)}{s^2 + 20s + (100+K)} = \frac{100A}{100+K}}$$

b) Sources of disturbance could be:

- wind
- contact with the tree
- forces from picking fruit
- effects of mass of picked fruit

Problem E4.4

E4.4 A magnetic disk drive requires a motor to position a read/write head over tracks of data on a spinning disk, as shown in Figure E4.4. The motor and head may be represented by the transfer function

$$G(s) = \frac{100}{s(\tau s + 1)},$$

where $\tau = 0.001$ s. The controller takes the difference of the actual and desired positions and generates an error. This error is multiplied by an amplifier K. (a) What is the steady-state position error for a step change in the desired input? (b) Calculate the required K in order to yield a steady-state error of 0.1 mm for a ramp input of 10 cm/s.

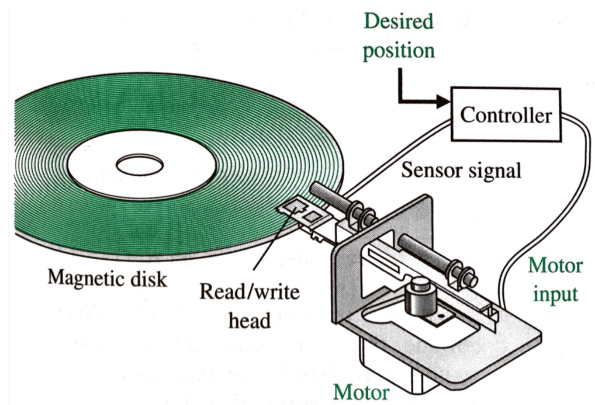
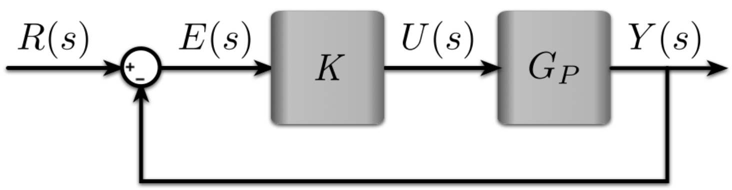


FIGURE E4.4 Disk drive control.

The block diagram of the system described would look something like:



We need to find the transfer function representing the error of the closed-loop system

$$E = R - Y = R - GU = R - GKE$$

$$(1 + KG)E = R$$

$$E = \frac{1}{1+KG} R = \left[\frac{1}{1 + \frac{100K}{s(\tau s+1)}}\right] R = \left[\frac{s(\tau s+1)}{s(\tau s+1)+100K}\right] R = \left[\frac{\tau s^2 + s}{\tau s^2 + s + 100K}\right] R$$

a) For a step change $r(t)$, $R(s) = \frac{A}{s}$, where A is the amplitude of the step

Use the Final Value Theorem to calculate the steady-state error

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$$

$$\boxed{\lim_{s\to 0} sE(s) = \lim_{s\to 0} s\left[\frac{\tau s^2 + s}{\tau s^2 + s + 100K}\right]\left[\frac{A}{s}\right] = 0}$$

Problem E4.4 (cont.)

b) A ramp of 10cm/s is $r(t) = 10t \rightarrow R(s) = \frac{10}{s^2}$

Again using the Final Value theorem

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\left[\frac{\tau s^2 + s}{\tau s^2 + s + 100K}\right]\left[\frac{10}{s^2}\right]$$

$$\lim_{s\to 0} \frac{10(\tau s^2 + s)}{\tau s^3 + s^2 + 100Ks} = \frac{10}{100K}$$ ← cm, since we wrote the ramp in cm

We want $\frac{10}{100K} < 0.1\text{mm} = \frac{10}{100K} < 0.01 \text{ cm} \rightarrow \boxed{K > 10}$

Problem E4.12

E4.12 In Figure E4.12, consider the closed-loop system with measurement noise $N(s)$, where

$$G(s) = \frac{100}{s+100}, \quad G_c(s) = K_1, \quad \text{and} \quad H(s) = \frac{K_2}{s+5}.$$

In the following analysis, the tracking error is defined to be $E(s) = R(s) - Y(s)$:

(a) Compute the transfer function $T(s) = Y(s)/R(s)$ and determine the steady-state tracking error due to a unit step response, that is, let $R(s) = 1/s$ and assume that $N(s) = 0$.

(b) Compute the transfer function $Y(s)/N(s)$ and determine the steady-state tracking error due to a unit step disturbance response, that is, let $N(s) = 1/s$ and assume that $R(s) = 0$. Remember, in this case, the desired output is zero.

(c) If the goal is to track the input while rejecting the measurement noise (in other words, while minimizing the effect of $N(s)$ on the output), how would you select the parameters K_1 and K_2?

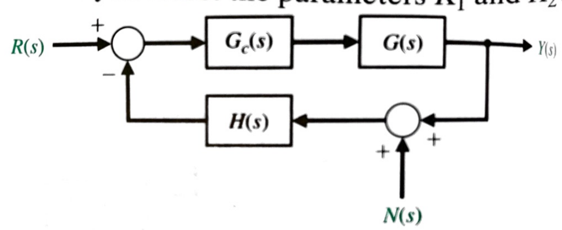


FIGURE E4.12 Closed-loop system with nonunity feedback and measurement noise.

a) $Y = G[G_c E] = GG_c[R - H(Y+N)]$

$= GG_cR - GG_cHY - GG_cHN$

$[1+G_cGH]Y = G_cGR - G_cGHN$

$$Y = \frac{G_cG}{1+G_cGH}R - \frac{G_cGH}{1+G_cGH}N$$

Let $N=0 \rightarrow Y = \frac{G_cG}{1+G_cGH}R \rightarrow$

$$\boxed{\frac{Y}{R} = \frac{G_cG}{1+G_cGH}}$$

$$E = R - Y = \left[1 - \frac{G_cG}{1+G_cGH}\right]R$$

For a step input $R(s) = \frac{1}{s}$

The steady state error is

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = s\left[\left[1 - \frac{G_cG}{1+G_cGH}\right]\left(\frac{1}{s}\right)\right] = \frac{1+G_cGH-G_cG}{1+G_cGH}$$

$$= \frac{1 + K_1\left(\frac{100}{s+100}\right)\left(\frac{K_2}{s+5}\right) - K_1\left(\frac{100}{s+100}\right)}{1 + K_1\left(\frac{100}{s+100}\right)\left(\frac{K_2}{s+5}\right)} = \frac{(s+100)(s+5) + 100K_1K_2 - 100K_1(s+5)}{(s+100)(s+5) + 100K_1K_2}$$

$$= \frac{500 + 100K_1K_2 - 500K_1}{500 + 100K_1K_2} = \boxed{\frac{5 + K_1K_2 - 5K_1}{5 + K_1K_2}}$$

Problem E4.12 (cont.)

b) $R(s) = 0$ and $N(s) = 1/s$

$$Y = \cancel{\frac{G_cG}{1+G_cGH}R} - \frac{G_cGH}{1+G_cGH}N \quad \text{so} \quad \boxed{\frac{Y}{N} = \frac{-G_cGH}{1+G_cGH}}$$

$$E = R - Y = 0 - Y = -Y$$

$$= \frac{+G_cGH}{1+G_cGH}N$$

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\left[\frac{+G_cGH}{1+G_cGH}\right]\left(\frac{1}{s}\right) = \lim_{s\to 0}\frac{G_cGH}{1+G_cGH}$$

$$= \lim_{s\to 0}\frac{K_1\left(\frac{100}{s+100}\right)\left(\frac{K_2}{s+5}\right)}{1+K_1\left(\frac{100}{s+100}\right)\left(\frac{K_2}{s+5}\right)} = \lim_{s\to 0}\frac{100K_1K_2}{(s+100)(s+5)+100K_1K_2}$$

$$= \frac{100K_1K_2}{500+100K_1K_2} = \boxed{\frac{K_1K_2}{5+K_1K_2}}$$

c) The steady state tracking error to the reference command was

$$e_{ss} = \frac{5+K_1K_2-5K_1}{5+K_1K_2}$$

A large K_1K_2 results in low error (good tracking)

While the steady-state error to noise was

$$e_{ss} = \frac{K_1K_2}{5+K_1K_2}$$

But, a small K_1K_2 best rejects noise

Problem P4.2

P4.2 It is important to ensure passenger comfort on ships by stabilizing the ship's oscillations due to waves [13]. Most ship stabilization systems use fins or hydrofoils projecting into the water to generate a stabilization torque on the ship. A simple diagram of a ship stabilization system is shown in Figure P4.2. The rolling motion of a ship can be regarded as an oscillating pendulum with a deviation from the vertical of $\theta(t)$ degrees and a typical period of 3 s. The transfer function of a typical ship is

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2},$$

where $\omega_n = 3.5$ rad/s and $\zeta = 0.25$. With this low damping factor ζ, the oscillations continue for several cycles, and the rolling amplitude can reach 18° for the expected amplitude of waves in a normal sea. Determine and compare the open-loop and closed-loop system for (a) sensitivity to changes in the actuator constant K_a and the roll sensor K_1, and (b) the ability to reduce the effects of step disturbances of the waves. Note that the desired roll $\theta_d(s)$ is zero degrees.

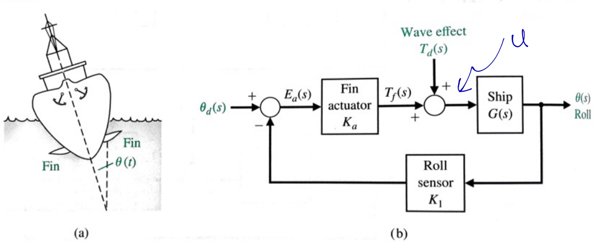


FIGURE P4.2 Ship stabilization system. The effect of the waves is a torque $T_d(s)$ on the ship.

Let's begin by writing the TF from

$\theta_d \to \theta$

$$\theta = GU = G(T_f + T_d)$$
$$= G(K_a E_a + T_d)$$
$$\theta = G(K_a(\theta_d - K_1\theta) + T_d)$$
$$\theta = K_a G\theta_d - K_1 K_a G\theta + GT_d$$
$$(1 + K_1K_aG)\theta = K_aG\theta_d + GT_d$$
$$\theta = \frac{K_aG}{1+K_1K_aG}\theta_d + \frac{G}{1+K_1K_aG}T_d$$

This defines the closed loop TF to either the input θ_d or disturbance T_d

The open-loop TF is (just ignore the FB)

$$\theta = G(T_d + T_f)$$
$$= G(T_d + K_a\theta_d)$$
$$\theta = K_aG\theta_d + GT_d$$

a) Let $T_d = 0$... the open-loop sensitivity for

K_1 is undefined, since it's not part of the open-loop system

$$\frac{\theta}{\theta_d} = T(s) = K_aG \rightarrow S^T_{K_a} = \frac{\partial T}{\partial K_a}\cdot\frac{K_a}{T} = G\cdot\frac{K_a}{K_aG} = 1$$

Open-loop

Problem P4.2 (cont.)

For the closed loop system

$$\frac{\Theta}{\Theta_d} = T(s) = \frac{K_a G}{1 + K_1 K_a G}$$

$$S_{K_a}^T = \frac{\partial T}{\partial K_a} \cdot \frac{K_a}{T} = \left[\frac{G}{1+K_1K_aG} + \frac{(K_aG)(-K_1G)}{(1+K_1K_aG)^2} \right] \cdot \frac{K_a}{T}$$

$$= \left[\frac{G(1+K_1K_aG) - K_1K_aG^2}{(1+K_1K_aG)^2} \right] \cdot \frac{K_a}{T}$$

$$= \left[\frac{\cancel{G}}{(1+K_1K_aG)^{\cancel{2}}} \right] \left[\frac{\cancel{K_a}\,\cancel{(1+K_1K_aG)}}{\cancel{K_a}\,\cancel{G}} \right]$$

$$\boxed{S_{K_a}^T = \frac{1}{1+K_1K_aG}}$$

$$S_{K_1}^T = \frac{\partial T}{\partial K_1} \cdot \frac{K_1}{T} = \left[\frac{\cancel{(K_aG)}(-K_aG)}{(1+K_1K_aG)^{\cancel{2}}} \right] \left[\frac{K_1\,\cancel{(1+K_1K_aG)}}{\cancel{K_aG}} \right]$$

$$\boxed{S_{K_1}^T = \frac{-K_1K_aG}{1+K_1K_aG}}$$

Problem P4.2 (cont.)

b) For a desired roll angle of 0 the TF is

The open-loop TF is

$$\Theta = G T_d \rightarrow E = \underbrace{\Theta_d}_{=0} - \Theta \Rightarrow \text{so } E = -\Theta$$

A step disturbance of amplitude A has the form $T_d(s) = \frac{A}{s}$

Use the Final Value Theorem to get an idea about the systems ability to recover from such disturbances

$$\boxed{\lim_{t\to\infty} -\Theta(t) = \lim_{s\to 0} -s\Theta(s) = sG\frac{A}{s} = GA = \left[\frac{-\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\right]A = -A}$$

Open-loop can not reject the disturbance

For the closed-loop system

$$\Theta = \frac{K_a G}{1+K_1K_aG}\cancelto{0}{\Theta_d} + \frac{G}{1+K_1K_aG}T_d$$

A step disturbance of amplitude A has the form $T_d(s) = \frac{A}{s}$

Use the Final Value Theorem to get an idea about the systems ability to recover from such disturbances

$$\lim_{t\to\infty} -\Theta(t) = \lim_{s\to 0} -s\Theta(s) = s\left[\frac{-G}{1+K_1K_aG}\left(\frac{A}{s}\right)\right] = \frac{-AG}{1+K_1K_aG}$$

$$\boxed{\lim_{s\to 0}\frac{-A\left(\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\right)}{1+K_1K_a\left(\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\right)} = \frac{-A\omega_n^2}{(s^2+2\zeta\omega_n s+\omega_n^2)+K_1K_a\omega_n^2} = \frac{-A\omega_n^2}{(1+K_1K_a)\omega_n^2} = \frac{-A}{1+K_1K_a}}$$

The larger we can make K_1 and K_a, the smaller the effects of the disturbance will be.

Problem E5.2

E5.2 The engine, body, and tires of a racing vehicle affect the acceleration and speed attainable [9]. The speed control of the car is represented by the model shown in Figure E5.2. (a) Calculate the steady-state error of the car to a step command in speed. (b) Calculate percent overshoot of the speed to a step command.

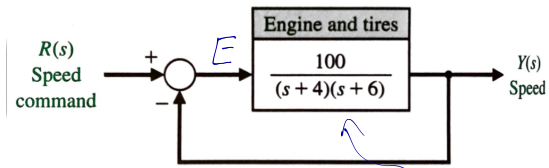


FIGURE E5.2 Racing car speed control.

Call this G

a) We need to calculate the error transfer function for the closed-loop system

$$E = R - Y = R - GE$$

$$(1+G)E = R \longrightarrow E = \frac{1}{1+G}R = \frac{1}{1+\frac{100}{(s+4)(s+6)}} = \frac{(s+4)(s+6)}{(s+4)(s+6)+100}R$$

A step input of amplitude A in $r(t) \rightarrow R(s) = \frac{A}{s}$

Use the Final Value Theorem to find the steady-state error

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$$

$$\boxed{\lim_{s\to 0} sE(s) = s\left[\frac{(s+4)(s+6)}{(s+4)(s+6)+100}\left(\frac{A}{s}\right)\right] = \frac{24A}{124} = \frac{6A}{31}}$$

b) For overshoot, we need the closed-loop TF from $R \to Y$

$$Y = GE = \frac{G}{1+G}R$$

$$Y = \left[\frac{100}{\cancel{(s+4)}\cancel{(s+6)}}\right]\left[\frac{\cancel{(s+4)}\cancel{(s+6)}}{(s+4)(s+6)+100}R\right] = \frac{100}{(s+4)(s+6)+100}R = \frac{100}{s^2+10s+124}R$$

$$\frac{Y}{R} = \frac{100}{s^2+10s+124}$$

This is a 2^{nd}-order system, so we can use the overshoot we calculated for such systems

We found that

$$\text{Percent overshoot} = 100\exp\left[-\zeta\pi/\sqrt{1-\zeta^2}\right]$$

To find ζ, match terms in the denominator of the TF to the 2^{nd}-order form

$2\zeta\omega_n = 10$ and $\omega_n^2 = 124$

$\zeta\omega_n = 5$ $\quad\omega_n \doteq 11.14\ \text{rad/s}$

ζ

Problem E5.2 (cont.)

We found that

$$\text{Percent overshoot} = 100 \exp\left[-\zeta\pi / \sqrt{1-\zeta^2}\right]$$

To find ζ, match terms in the denominator of the TF to the 2nd-order form

$$2\zeta\omega_n = 10 \qquad \text{and} \qquad \omega_n^2 = 124$$

$$\zeta\omega_n = 5 \qquad\qquad \omega_n \cong 11.14 \text{ rad/s}$$

$$\zeta \cong \frac{5}{11.14} \approx 0.45$$

So, percent overshoot is

$$\boxed{PO = 100 \exp\left[-0.45\pi / \sqrt{1-0.45^2}\right] \cong 20.5\%}$$

Problem P5.3

P5.3 A laser beam can be used to weld, drill, etch, cut, and mark metals, as shown in Figure P5.3(a) [14]. Assume we have a work requirement for an accurate laser to mark a parabolic path with a closed-loop control system, as shown in Figure P5.3(b). Calculate the necessary gain to result in a steady-state error of 5 mm for $r(t) = t^2$ cm.

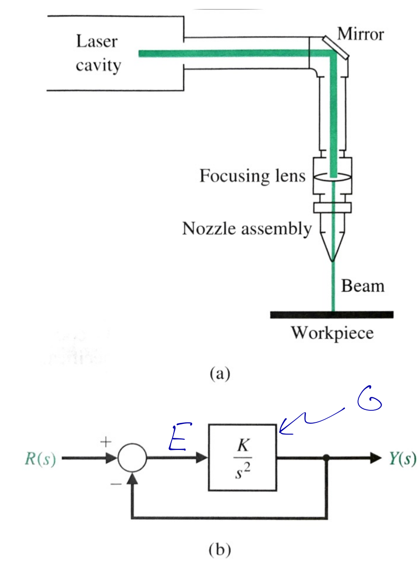


FIGURE P5.3 Laser beam control.

Find the TF that represents the error of the closed-loop system.

$$E = R - Y = R - GE$$

$$[1+G]E = R \rightarrow E = \frac{1}{1+G} R = \frac{s^2}{s^2+K} R$$

To find the steady-state error use the Final Value Theorem

$$\lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s)$$

For $r(t) = t^2 \rightarrow R(s) = \frac{1}{s^3}$ ← Look up in Laplace Table if you don't remember

$$\lim_{s \to 0} s\left[\left(\frac{s^2}{s^2+K}\right)\left(\frac{1}{s^3}\right)\right] = \lim_{s \to 0} \frac{1}{s^2+K} = \frac{1}{K}$$

← We want this < 5mm (0.5cm, since input is cm)

$$\frac{1}{K} < 0.5 \rightarrow \boxed{K > 2}$$