MCHE 474: Control Systems
Fall 2017 – Homework 2

Assigned: Thursday, September 8th
Due: Friday, September 15th, 5pm

Assignment: From *Modern Control Systems (13th Edition)* by Richard Dorf and Robert Bishop, solve problems:
E2.4, E2.8, E2.25, E2.27, E3.2, E3.19, P3.15

Submission: Emailed *single* pdf document:

- to `joshua.vaughan@louisiana.edu`
- with subject line `CLID-MCHE474-HW2,` where `CLID` is replaced with your CLID
- and that has a *single* pdf attached with filename `CLID-MCHE474-HW2,` where `CLID` is replaced with your CLID
- *Note:* Submissions with incorrect filenames or submitted as multiple images/pdfs will be rejected.

E2.4

E2.4 A laser printer uses a laser beam to print copy rapidly for a computer. The laser is positioned by a control input $r(t)$, so that we have

$$Y(s) = \frac{6(s+50)}{s^2+40s+300}R(s).$$

The input $r(t)$ represents the desired position of the laser beam.
(a) If $r(t)$ is a unit step input, find the output $y(t)$.
(b) What is the final value of $y(t)$?

a.) If $r(t)$ is a unit step, $R(s) = \frac{1}{s}$, so

$$Y(s) = \frac{6(s+50)}{s^2+40s+300}\left(\frac{1}{s}\right)$$

$$Y(s) = \frac{6(s+50)}{s(s+30)(s+10)} = \frac{a_1}{s} + \frac{a_2}{s+10} + \frac{a_3}{s+30}$$

$$a_1 = \left(\cancel{s}\left[\frac{6(s+50)}{\cancel{s}(s+30)(s+10)}\right]\right)\Bigg|_{s=0} = \frac{6(50)}{(30)(10)} = \frac{300}{300} = 1$$

$$a_2 = \left(\cancel{(s+10)}\left[\frac{6(s+50)}{s(s+30)\cancel{(s+10)}}\right]\right)\Bigg|_{s=-10} = \frac{6(40)}{-10(20)} = \frac{240}{-200} = -1.2$$

$$a_3 = \left(\cancel{(s+30)}\left[\frac{6(s+50)}{s\cancel{(s+30)}(s+10)}\right]\right)\Bigg|_{s=-30} = \frac{6(20)}{(-30)(-20)} = \frac{120}{600} = \frac{1}{5} = 0.2$$

$$Y(s) = \frac{1}{s} - \frac{1.2}{s+10} + \frac{0.2}{s+30}$$

So

$$\boxed{y(t) = 1 - 1.2e^{-10t} + 0.2e^{-30t}}$$

b.) Just use the Final Value theorem:

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} \frac{\cancel{s}\,6(s+50)}{\cancel{s}(s+30)(s+10)} = \frac{300}{300} = 1$$

E2.8

E2.8 A control engineer, N. Minorsky, designed an innovative ship steering system in the 1930s for the U.S. Navy. The system is represented by the block diagram shown in Figure E2.8, where $Y(s)$ is the ship's course, $R(s)$ is the desired course, and $A(s)$ is the rudder angle [16]. Find the transfer function $Y(s)/R(s)$.

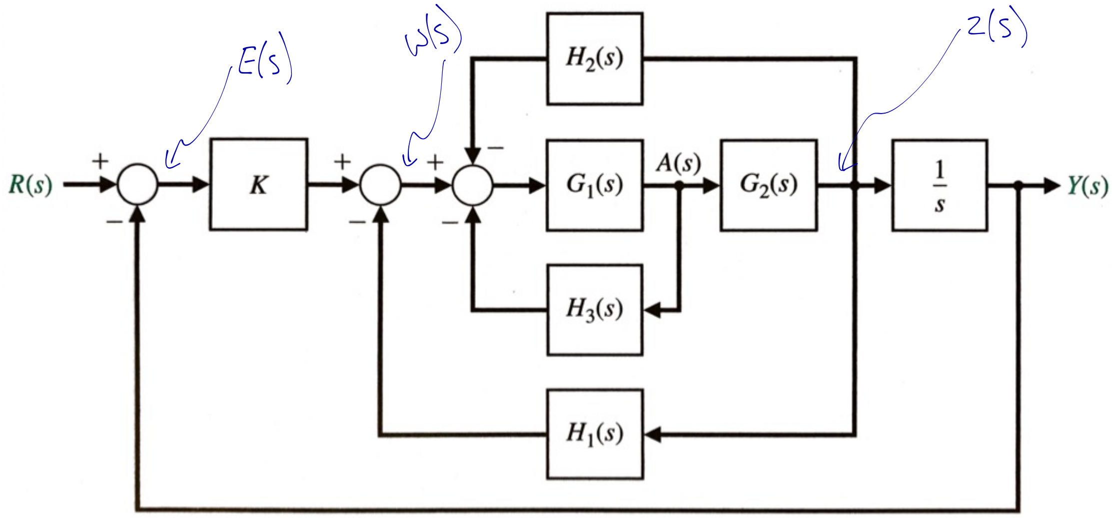


FIGURE E2.8 Ship steering system.

$$Y(s) = \frac{1}{s} Z(s) = \frac{1}{s}\left[G_2(s) A(s)\right]$$ ← I will drop the (s) after this line

$$A(s) = G_1(W - H_2 Z - H_3 A) \longrightarrow A = \frac{G_1 W - G_1 H_2 Z}{1 + G_1 H_3}$$

$$W = KE - H_1 Z = K(R - Y) - H_1 Z$$

$$A = \frac{1}{G_2} Z = \frac{s}{G_2} Y$$

$$Z = sY$$ so

$$Y = \frac{1}{s}\left[G_2 G_1 (W - H_2 Z - H_3 A)\right]$$

$$Y = \frac{1}{s}\left[G_2 G_1 \left(W - sH_2 Y - H_3\left(\frac{1}{G_2} Y\right)\right)\right]$$

$$= \frac{1}{s}\left[G_2 G_1 \left(KR - KY - sH_1 Y - sH_2 Y - \frac{H_3}{G_2} Y\right)\right]$$

$$= \frac{1}{s} G_1 G_2 KR - \frac{1}{s} G_1 G_2 KY - G_1 G_2 H_1 Y - G_1 G_2 H_2 Y - G_1 H_3 Y$$

E2.8 (cont.)

$$Y(s) = \frac{1}{s} G_1 G_2 K R - \frac{1}{s} G_1 G_2 K Y - G_1 G_2 H_1 Y - G_1 G_2 H_2 Y - G_1 H_3 Y$$

$$sY = G_1 G_2 K R - G_1 G_2 K Y - s G_1 G_2 H_1 Y - s G_1 G_2 H_2 Y - s G_1 H_3 Y$$

$$\left(s + G_1 G_2 K + s G_1 G_2 H_1 + s G_1 G_2 H_2 + s G_1 H_3 \right) Y = G_1 G_2 K R$$

$$\boxed{\frac{Y}{R} = \frac{K G_1 G_2}{s + G_1 G_2 K + s G_1 G_2 H_1 + s G_1 G_2 H_2 + s G_1 H_3}}$$

E2.25

E2.25 The block diagram of a system is shown in Figure E2.25. Determine the transfer function $T(s) = Y(s)/R(s)$.

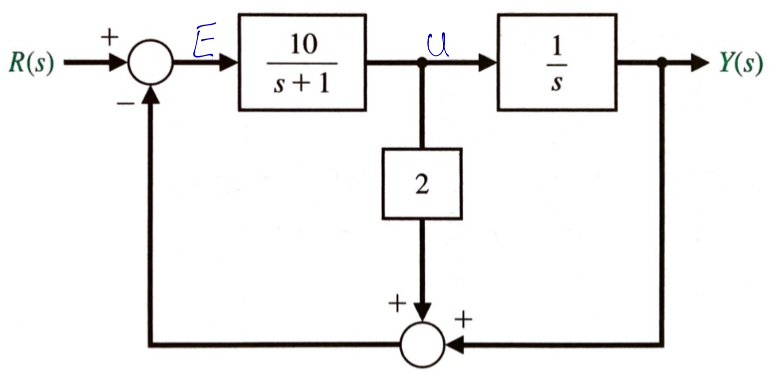


FIGURE E2.25 Multiloop feedback system.

$Y = \frac{1}{s}u \qquad u = sY$

$u = \frac{10}{s+1}E$

$E = R - (Y + 2u) = R - Y - 2u = R - Y - s2Y$

$Y = \frac{1}{s}\left[\frac{10}{s+1}\right]\left[R - Y - s2Y\right]$

$(s^2 + s)Y = 10R - 10(2s+1)Y$

$(s^2 + s + 20s + 10)Y = 10R$

$$\boxed{\frac{Y}{R} = \frac{10}{s^2 + 21s + 10}}$$

E2.27

E2.27 Find the transfer function $Y(s)/T_d(s)$ for the system shown in Figure E2.27.

Answer: $$\frac{Y(s)}{T_d(s)} = \frac{G_2(s)}{1 + G_1(s)G_2(s)H(s)}$$

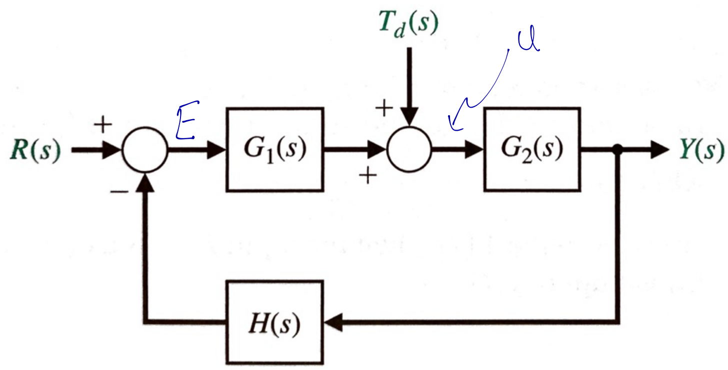


FIGURE E2.27 System with disturbance.

$$Y = G_2 U$$

$$U = T_d + G_1 E$$

$$E = R - HY$$

$$U = T_d + G_1 R - G_1 H Y$$

$$Y = G_2 (T_d + G_1 R - G_1 H Y)$$

$$(1 + G_1 G_2 H) Y = G_2 T_d + G_1 G_2 R$$

$$Y = \frac{G_2}{1 + G_1 G_2 H} T_d + \frac{G_1 G_2}{1 + G_1 G_2 H} R$$

For TF from $T_d \to Y$, assume $R = 0$

$$Y = \frac{G_2}{1 + G_1 G_2 H} T_d$$

$$\boxed{\frac{Y}{T_d} = \frac{G_2}{1 + G_1 G_2 H}}$$

E3.2

E3.2 A robot-arm drive system for one joint can be represented by the differential equation [8]

$$\frac{dv(t)}{dt} = -k_1 v(t) - k_2 y(t) + k_3 i(t),$$

where $v(t)$ = velocity, $y(t)$ = position, and $i(t)$ is the control-motor current. Put the equations in state variable form and set up the matrix form for $k_1 = k_2 = 1$.

Define $v(t) = \dot{y}$ and $\frac{dv}{dt} = \ddot{y}$

$$\ddot{y} = -k_1 \dot{y} - k_2 y + k_3 i$$

Define the state vector $\bar{x} = \begin{bmatrix} y \\ \dot{y} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\dot{\bar{x}} = \begin{bmatrix} \dot{y} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} \dot{y} \\ -k_1 \dot{y} - k_2 y + k_3 i \end{bmatrix} = \begin{bmatrix} x_2 \\ -k_1 x_2 - k_2 x_1 + k_3 i \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k_2 & -k_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ k_3 \end{bmatrix} i$$

$$\boxed{\dot{\bar{x}} = \begin{bmatrix} 0 & 1 \\ -k_2 & -k_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ k_3 \end{bmatrix} i}$$

For $k_1 = k_2 = 1$

$$\dot{\bar{x}} = A\bar{x} + B\bar{u}$$

$$\boxed{\dot{\bar{x}} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\bar{x}} + \underbrace{\begin{bmatrix} 0 \\ k_3 \end{bmatrix}}_{B} \underbrace{i}_{\bar{u}}}$$

$$\dot{\bar{x}} = A \quad \bar{x} + B \; \bar{u}$$

E3.19

E3.19 A single-input, single-output system has the matrix equations

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ -3 & -5 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u(t)$$

and

$$y(t) = [3 \quad 0]\mathbf{x}(t).$$

Determine the transfer function $G(s) = Y(s)/U(s)$.

We can solve this by:

1) Following procedure from section 3.6 of the book <u>or</u>

2) Working back to the ODE and getting TF from that.

Let's start with method (2)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -3x_1 - 5x_2 + u$$

$$y = [3 \quad 0]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 3x_1$$

$$x_1 = \frac{1}{3}y \qquad x_2 = \dot{x}_1 = \frac{1}{3}\dot{y} \qquad \dot{x}_2 = \frac{1}{3}\ddot{y}$$

$$\frac{1}{3}\ddot{y} = -3\left(\frac{1}{3}y\right) - 5\left(\frac{1}{3}\dot{y}\right) + u$$

$$\ddot{y} = -3y - 5\dot{y} + 3u$$

$$\ddot{y} + 5\dot{y} + 3y = 3u$$

$$(s^2 + 5s + 3)Y = 3U$$

$$\boxed{\frac{Y}{U} = \frac{3}{s^2 + 5s + 3}}$$

E3.19 (cont.)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -3 & -5 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u \qquad y = \underbrace{\begin{bmatrix} 3 & 0 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_{D} u$$

From Section 3.6, we know

$$G(s) = \frac{y}{u} = C\Phi(s)B + D \quad \text{where} \quad \Phi(s) = (sI - A)^{-1}$$

$$\Phi(s) = \left[s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -3 & -5 \end{bmatrix} \right]^{-1} = \begin{bmatrix} s & -1 \\ 3 & s+5 \end{bmatrix}^{-1}$$

$$= \frac{1}{\mathrm{Det}(sI-A)} \begin{bmatrix} s+5 & 1 \\ -3 & s \end{bmatrix} \qquad \mathrm{Det}(sI-A) = s(s+5) - (-3) = s^2 + 5s + 3$$

$$\Phi(s) = \begin{bmatrix} \frac{s+5}{s^2+5s+3} & \frac{1}{s^2+5s+3} \\ \frac{-3}{s^2+5s+3} & \frac{s}{s^2+5s+3} \end{bmatrix}$$

So

$$G(s) = \begin{bmatrix} 3 & 0 \end{bmatrix} \begin{bmatrix} \frac{s+5}{s^2+5s+3} & \frac{1}{s^2+5s+3} \\ \frac{-3}{s^2+5s+3} & \frac{s}{s^2+5s+3} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{s^2+5s+3} \\ \frac{s}{s^2+5s+3} \end{bmatrix}$$

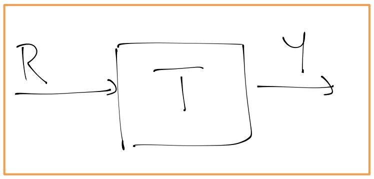
$$\boxed{G(s) = \frac{3}{s^2+5s+3}}$$

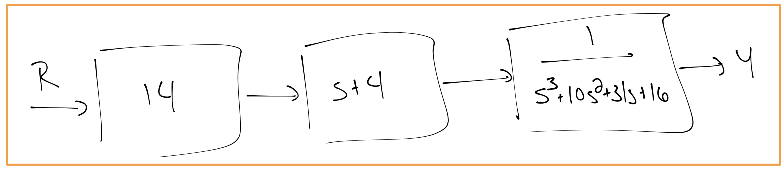
P3.15

P3.15 Obtain a block diagram and a state variable representation of this system.

$$\frac{Y(s)}{R(s)} = T(s) = \frac{14(s+4)}{s^3 + 10s^2 + 31s + 16}.$$

There are obviously many block diagrams that could represent this system. A few of the simplest are





To get to some more/complex ones, let's look at the TF. (and get the state variable representation in the process)

$$(s^3 + 10s^2 + 31s + 16)Y = 14(s+4)R = (14s + 56)R$$

Define $\bar{X} = [x_1 \quad x_2 \quad x_3]^T$ and

$$\dddot{y} + 10\ddot{y} + 31\dot{y} + 16y = 14\dot{r} + 56r \qquad x_3 = \ddot{y}$$

$$\dddot{y} = -10\ddot{y} - 31\dot{y} - 16y + 14\dot{r} + 56r$$

Define $\dot{x}_1 = x_2$ and $\dot{x}_2 = x_3$

$$\dot{\bar{X}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -16 & -31 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 56 & 14 \end{bmatrix} [r \quad \dot{r}]$$

$$y = [0 \quad 0 \quad 1]\bar{X}$$