MCHE 474: Control Systems

Fall 2017 – Mid-Term 2

Thursday, November 9

Name: Answer Key **CLID:** ____________

Directions: Complete the attached problems making sure to clearly indicate your answer, show your work, and list any assumptions that you have made (with justification for them, if necessary). If you need extra space for any question, you may attach additional sheets of paper, which will be provided to you. No calculators are allowed.

Academic Honesty (just a reminder):

An essential rule in every class of the University is that all work for which a student will receive a grade or credit be entirely his or her own or be properly documented to indicate sources. When a student does not follow this rule, s/he is dishonest and s/he defeats the purpose of the course and undermines the goals of the University.

______ **Problem 1 – 40 Points**

For the block diagram shown in Figure 1:

a. Write the closed-loop transfer function.

b. Write the closed-loop natural frequency in term of K and K_1.

c. Write the closed-loop damping ratio in term of K and K_1.

d. Assuming $J = 1$ kg-m^2, find values of K and K_1 that would result in a maximum percent overshoot of 25% and a peak time of 2s.
Note: Just set up the equations needed to calculate the actual numerical values. Do so in a way such that all that would remain is plugging them into a calculator for solution.

e. What is the setting time for that choice of K and K_1?

f. What is the steady-state response of $y(t)$ to a unit step input in $r(t)$?

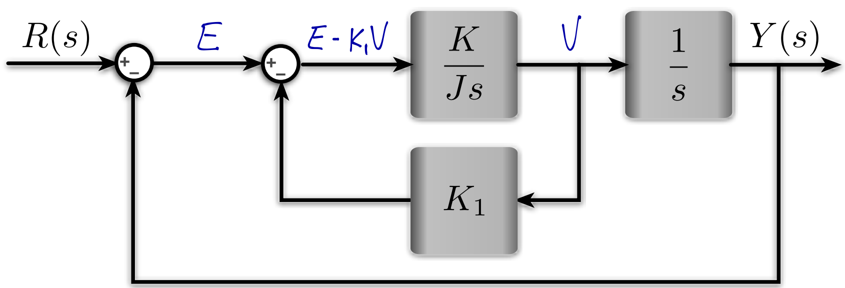


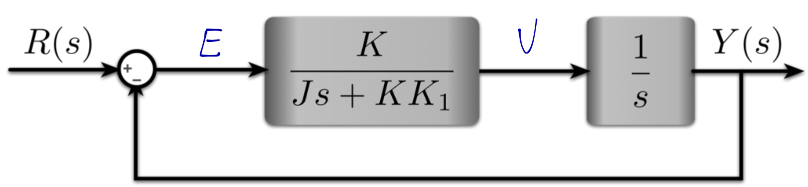
Figure 1: Block Diagram for Closed-loop System

a. $V = \frac{K}{Js}(E - K_1 V)$

$\left(1 + \frac{KK_1}{Js}\right) V = \frac{K}{Js} E$

$V = \left[\frac{K}{Js + KK_1}\right] E$

So the block diagram can be reduced to:



$Y = \frac{1}{s} V = \left[\frac{K}{Js^2 + KK_1 s}\right] E$

$Y = \left[\frac{K}{Js^2 + KK_1 s}\right](R - Y)$

$\left[1 + \frac{K}{Js^2 + KK_1 s}\right] Y = \frac{K}{Js^2 + KK_1 s} R$

$\left[Js^2 + KK_1 s + K\right] Y = KR \rightarrow \boxed{\frac{Y}{R} = \frac{K}{Js^2 + KK_1 s + K}}$

Problem 1 (cont.)

$$\frac{Y}{R} = \frac{K}{Js^2 + KK_1 s + K}$$

Divide num. + den by J to put in "standard" form

$$\frac{Y}{R} = \frac{K/J}{s^2 + \frac{KK_1}{J}s + \frac{K}{J}}$$

← Now match with $s^2 + 2\zeta\omega_n s + \omega_n^2$ for b + c

b. $\omega_n^2 = K/J \rightarrow \boxed{\omega_n = \sqrt{\frac{K}{J}}}$

c. $2\zeta\omega_n = \frac{KK_1}{J} \rightarrow \boxed{\zeta = \frac{KK_1}{2\omega_n J} = \frac{KK_1}{2J\sqrt{K/J}}}$

d. $\%\text{ overshoot} = 100 \exp\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right) = 25\%$

$$\exp\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right) = 0.25$$

$$\frac{-\zeta\pi}{\sqrt{1-\zeta^2}} = \ln(0.25)$$

← plug in solver to find $\zeta = 0.404$ is necessary for 25% overshoot } Not necessary to do on the exam

Peak time $T_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 2\,s$

need $\omega_d = \frac{\pi}{2}$ rad/s $\rightarrow \omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}}$ ← plug in ζ from above to find $\underbrace{\omega_n = 1.72 \text{ rad/s}}_{}$

number not necessary on the exam

Then, set $\omega_n = \sqrt{\frac{K}{J}} = \sqrt{K} = 1.72 \frac{\text{rad}}{\text{s}}$

So, $K = 2.95$

We then use $K_1 = \frac{2\zeta\omega_n J}{K}$ to find $K_1 = 0.47$ } numerical values not necessary on the exam

e. The settling time is

$T_s = \frac{4}{\zeta\omega_n}$ so, using the ζ and ω_n from part d., we find

$T_s = \frac{4}{(0.404)(1.72)} = 5.76\,s$ } numerical value not necessary on the exam

Problem 1 (cont.)

f. We found the CLTF to be:

$$\frac{Y}{R} = \frac{K}{Js^2 + KK_1 s + K} \qquad \text{so} \qquad Y = \left[\frac{K}{Js^2 + KK_1 s + K}\right] R$$

For a step input $R(s) = \frac{A}{s}$ where A is the step amplitude

To find the steady-state value, use the Final Value Theorem:

$$\lim_{t \to \infty} y(t) = \lim_{s \to 0} sY(s)$$

$$\lim_{s \to 0} sY(s) = \lim_{s \to 0} \cancel{s}\left[\left(\frac{K}{Js^2 + KK_1 s + K}\right)\left(\frac{A}{\cancel{s}}\right)\right] = \lim_{s \to 0} \frac{KA}{Js^2 + KK_1 s + K}$$

$$\boxed{\lim_{s \to 0} sY(s) = A}$$

______ Problem 2 – 40 Points

Consider a system represented by the block diagram in Figure 2, where:

$$G(s) = \frac{K}{s^2(s+1)} \quad \text{and} \quad H(s) = 1$$

a. Sketch the root locus for this system.

b. Is this system stable for any nonzero value of K? Why or why not? (*Hint:* Use the root locus you just drew to support your answer.

Now, add a zero to the transfer function of $G(s)$, such that

$$G(s) = \frac{K(s+a)}{s^2(s+1)} \quad \text{and} \quad H(s) = 1$$

where $0 \leq a < 1$.

d. Sketch the root locus for this new system.

e. Is this system *unstable* for any nonzero value of K? Why or why not? (*Hint:* Use the root locus you just drew to support your answer.

f. What is the steady-state error of this system in response to a ramp input?

g. Without calculating it directly, do you expect this system to have a nonzero, but finite steady-state error in response to a parabolic input? Why or why not?

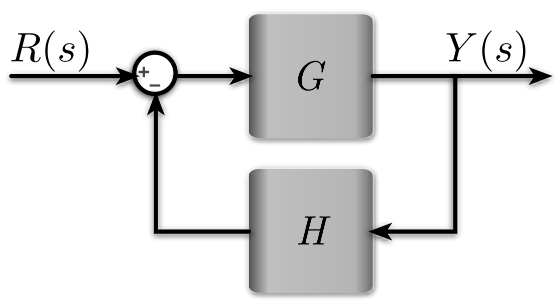


Figure 2: A Unity Feedback System

a. OL poles at $s = 0, 0, -1$

There will be 3 separate loci

$$\sigma_A = \frac{\sum p_j - \sum z_i}{n-m} = \frac{(0+0-1)}{3}$$

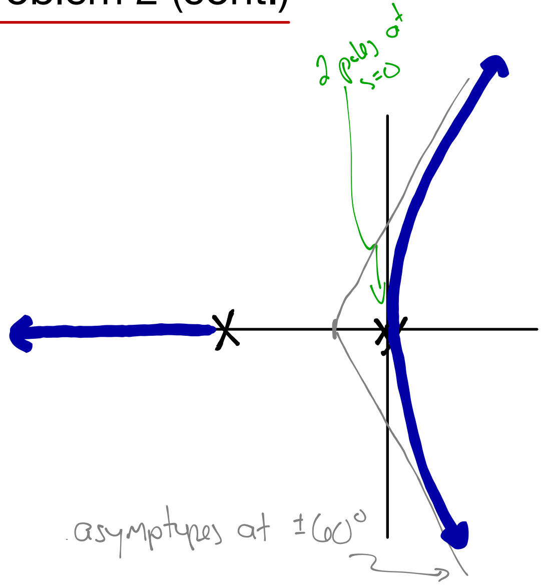
$$\phi_A = \left(\frac{2k+1}{n-m}\right)180^\circ \quad k=0,1,2 \quad = \left(\frac{2k+1}{3}\right)180^\circ \quad k=0,1,2 \rightarrow \pm 60^\circ, 180^\circ$$

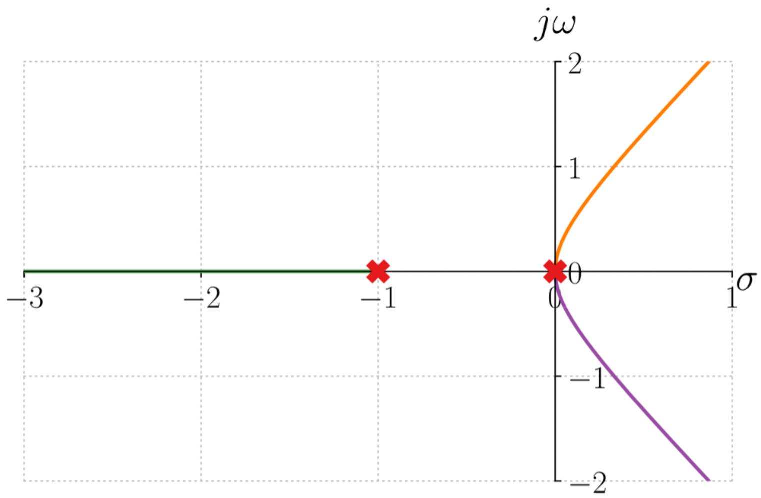
breakaway → at s where $\frac{dK}{ds} = 0$

$$K = -s^2(s+1) = -s^3 - s^2 \qquad \frac{dK}{ds} = -3s^2 - 2s = -s\left(s + \frac{2}{3}\right) \rightarrow s = 0,\ \cancel{s = -\frac{2}{3}}$$

not on locus so is the poin

Problem 2 (cont.)





b. For any nonzero value of K there are closed-loop poles in the RHP.

So, the system is unstable.

c. No c part. Sorry. I should do a better job of proofreading.

d. $G(s) = \dfrac{K(s+a)}{s^2(s+1)}$ $\quad 0 < a \leq 1$

α poles at $s = 0, 0, -1$

α zero at $-a$

3 separate loci

Asymptotes:

$$\sigma_A = \frac{(0+0-1)-(-a)}{2} = \frac{-1+a}{2} \Bigg\}$$ So, somewhere between 0 $(a=1)$ and $-1/2$ $(a=0)$

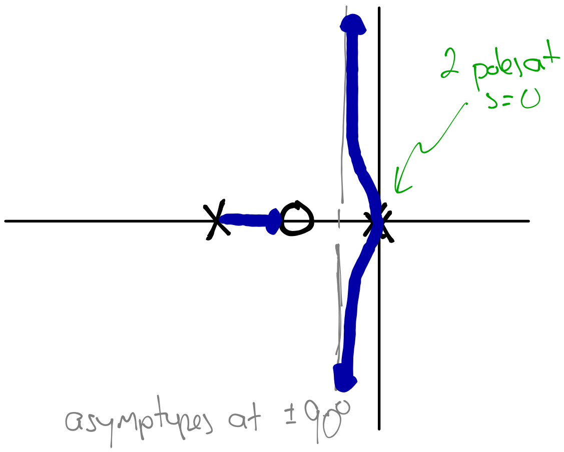
$$\phi_A = \left(\frac{2k+1}{2}\right)180^\circ \quad k = 0,1 \rightarrow \phi_A = \pm 90^\circ$$

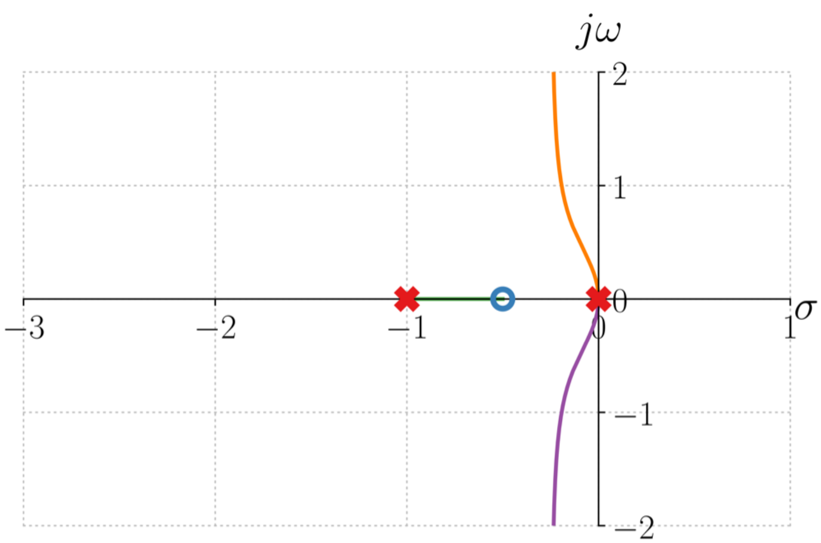
Problem 2 (cont.)

Breakaway Points:

$$K = \frac{-s^2(s+1)}{s+a} \qquad \frac{dK}{ds} = \frac{-s^3-s^2}{s+a} = \frac{-3s^2-2s}{s+a} + \frac{s^3+s^2}{(s+a)^2} = \frac{-s(-3s+1)(s+a) + s(s^2+1)}{(s+a)^2}$$

↑ s=0 is a possible solution & there is only 1 breakaway, so this must be it





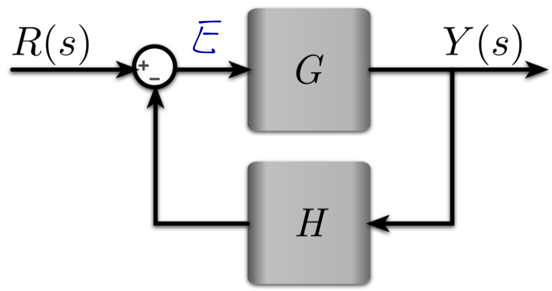
e. The closed-loop poles for this system are <u>never</u> in the RHP, so it is always stable.

Problem 2 (cont.)

f. This is a Type 2 system, so its steady-state error to a ramp input is 0.

If you didn't remember this, we need to:

Find $E(s)$ for the CLTF



$H=1$, so...

$E = R - Y = R - GE$

$(1+G)E = R$

$\frac{E}{R} = \frac{1}{1+G} =$

$G = \frac{K(s+a)}{s^2(s+1)}$ so $\frac{E}{R} = \frac{s^2(s+1)}{s^2(s+1)+K(s+a)}$

$E = \frac{s^2(s+1)}{s^2(s+1)+K(s+a)} R$

$R(s) = \frac{1}{s^2}$ for a unit ramp input so

$$E(s) = \frac{\cancel{s^2}(s+1)}{s^2(s+1)+K(s+a)}\left(\frac{1}{\cancel{s^2}}\right)$$

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \frac{s(s+1)}{s^2(s+1)+K(s+a)} = 0$$

g. This is a Type 2 system, so it will have a finite steady-state error in response to a parabolic input.

______ **Problem 3 – 20 Points**

For the system in the block diagram in Figure 3:

$$G_p(s) = \frac{10}{s(s+1)} \quad \text{and} \quad G_c(s) = \frac{s+a}{s+8}$$

where a is a positive constant.

a. What is the open-loop transfer function this system?

b. What is the closed-loop transfer function this system?

c. Sketch the root locus for this system for variation in parameter a.

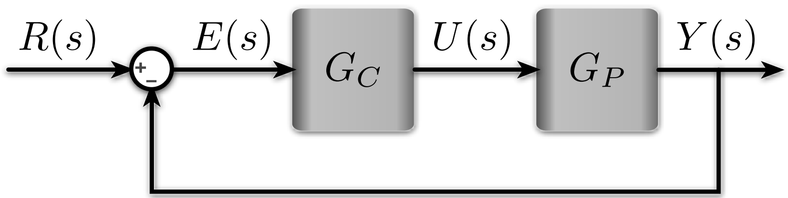


Figure 3: Block Diagram of a Feedback Control System

a. OLTF: $G_cG_p = \dfrac{10(s+a)}{s(s+1)(s+8)}$

b. CLTF: $\dfrac{G_cG_p}{1+G_cG_p} = \dfrac{10(s+a)}{s(s+1)(s+8)+10(s+a)}$

c. In its current form, a is not in the correct "place" for sketching a root locus based on its variation. So, we need to manipulate the char. eq. so that it is. The current char. eq. is

$$s(s+1)(s+8)+10(s+a)=0$$

Divide by all terms except 10a

$$\frac{s(s+1)(s+8)+10s+10a}{s(s+1)(s+8)+10s}=0 \rightarrow 1+\frac{10a}{s(s+1)(s+8)+10s}=0$$

$$\rightarrow 1+\underbrace{\frac{10a}{s(s^2+9s+18)}}_{}=0$$

Now, this in MC form we can use for a root locus

Problem 3 (cont.)

$$1 + \frac{10a}{s(s^2+9s+18)} = 0$$

Define $K = 10a$

$$\frac{K}{s(s^2+9s+18)}$$

OL poles at $s=0$, $s=-3$, $s=-6$

The locus will have 3 separate loci

Asymptotes:

$$\sigma_A = \frac{0-3-6}{3} = -3$$

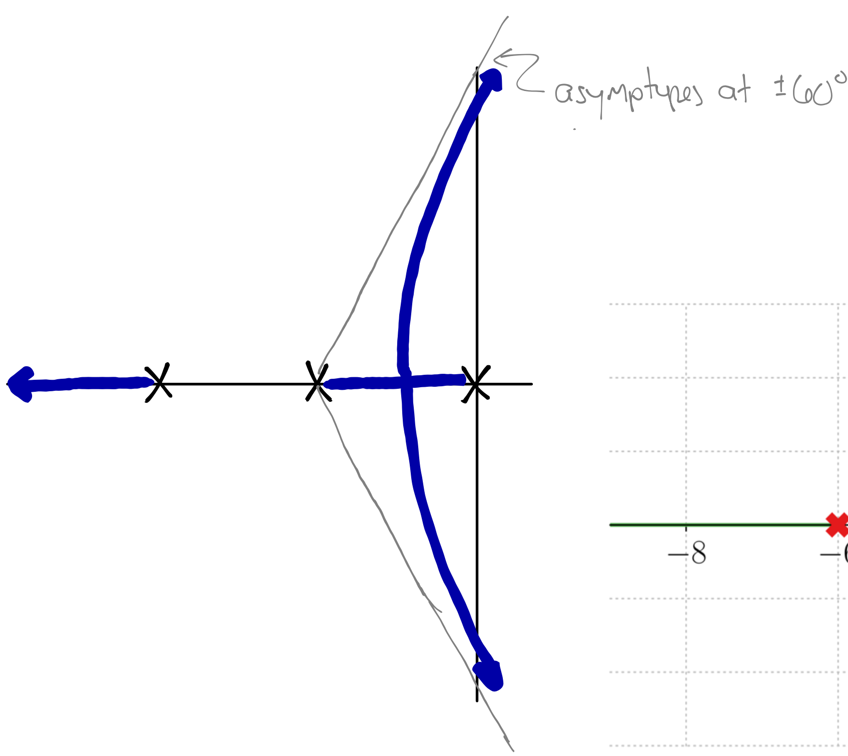
$$\phi_A = \left(\frac{2k+1}{3}\right)180^\circ \quad k=0,1,2 \rightarrow \phi_A = \pm 60^\circ, 180^\circ$$

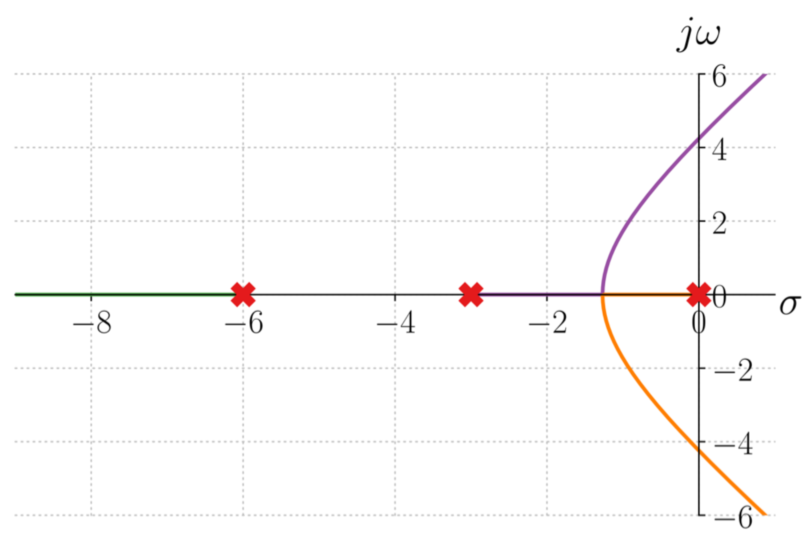
Breakaway Point

$$K = -s(s^2+9s+18) = -s^3-9s^2-18s$$

$$\frac{dK}{ds} = -3s^2-18s-18 = -3(\underbrace{s^2+6s+6}_{\text{Some point between } 0 \text{ and } -3})$$

(actual $s=-1.27$ and $s=-4.73$)





Possibly Useful Equations

$$\bar{f} = m\bar{a}$$

$$I_0\bar{\alpha} = \sum \bar{M}_0$$

$$\sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b)$$

$$\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b)$$

$$e^{\pm i\omega t} = \cos(\omega t) \pm i\sin(\omega t)$$

$$x(t) = ae^{i\omega_n t} + be^{-i\omega_n t}$$

$$x(t) = a\cos\omega_n t + b\sin\omega_n t$$

$$x(t) = e^{-\zeta\omega_n t}\left[a\cos(\omega_d t) + b\sin(\omega_d t)\right]$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\int u\ dv = uv - \int v\ du$$

$$\delta_{oc}V = \forall \sum$$

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$$
$$\mathbf{y} = C\mathbf{x} + D\mathbf{u}$$

$$\mathbf{x} = \mathbf{\Phi}(t)\mathbf{x}(0) + \int_0^t \mathbf{\Phi}(t-\tau)\mathbf{B}\mathbf{u}(\tau)d\tau$$

$$\mathbf{X}(s) = \left[s\mathbf{I} - \mathbf{A}\right]^{-1}\mathbf{x}(0) + \left[s\mathbf{I} - \mathbf{A}\right]^{-1}\mathbf{B}\mathbf{U}(s)$$

$$S_\alpha^T = S_G^T S_\alpha^G$$

$$T_s = \frac{4}{\zeta\omega_n}$$

$$T_r \approx \frac{2.16\zeta + 0.60}{\omega_n}, \quad 0.3 \leq \zeta \leq 0.8$$

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

$$M_{P_t} = 1 + e^{-\zeta\pi/\sqrt{1-\zeta^2}}$$

$$\%\text{Overshoot} = 100e^{-\zeta\pi/\sqrt{1-\zeta^2}}$$

$$V(\omega,\zeta) = e^{-\zeta\omega t_n}\sqrt{[C(\omega,\zeta)]^2 + [S(\omega,\zeta)]^2}$$

Table 1: Laplace Transform Pairs

$f(t)$	$F(s)$
$f(t) = A,\ \forall t > 0$	$\frac{A}{s}$
$f(t) = At,\ \forall t > 0$	$\frac{A}{s^2}$
$\delta(t)$	1
e^{-at}	$\frac{1}{s+a}$
$\sin\omega t$	$\frac{\omega}{s+\omega^2}$
$\cos\omega t$	$\frac{s}{s+\omega^2}$
$e^{-at}\sin\omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos\omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$
$\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}\right),\ \zeta < 1$	$\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$
$1 - \frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2} + \phi\right),\ \phi = \cos^{-1}\zeta,\ \zeta < 1$	$\frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}$