# MCHE 474: Control Systems

Fall 2017 – Mid-Term 1

Tuesday, October 3

**Name:** Answer Key **CLID:** _____

**Directions:** Complete the attached problems making sure to clearly indicate your answer, show your work, and list any assumptions that you have made (with justification for them, if necessary). If you need extra space for any question, you may attach additional sheets of paper, which will be provided to you. No calculators are allowed.

**Academic Honesty (just a reminder):**

An essential rule in every class of the University is that all work for which a student will receive a grade or credit be entirely his or her own or be properly documented to indicate sources. When a student does not follow this rule, s/he is dishonest and s/he defeats the purpose of the course and undermines the goals of the University.

_____ **Problem 1 – 25 Points**

The system in Figure 1 consists of mass, $m$, connected to ground through a spring of spring constant $k$ and a damper of damping coefficient $c$. Until time $t = 0$, the mass is held so that gravitational forces have no net effect on the system. At time $t = 0$ the mass is released, so that gravitational forces *do* affect the system.

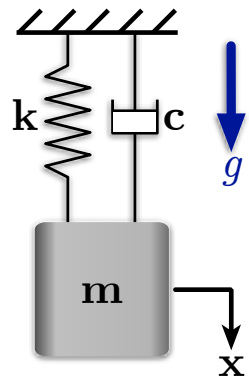



**Figure 1: A Mass-Spring-Damper System**

a. Write the equations of motion for this system.

b. What is the natural frequency?

c. What is the damping ratio?

If you were unable to solve parts a. to c. of this problem, assume the equations of motion have the form $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = u$ in the next two parts of this question. If you were able to solve parts a. to c. of this problem, use your answer from there.

d. What is the transfer function relating the position of the mass to the gravitational force?

e. What is the time response, $x(t)$, of this system? (*Hint:* Consider the release of the mass at $t = 0$ to be a step change in gravitational force.)

f. What is the steady-state position of the mass?

a.


$$m\ddot{x} = -F_{sp} - F_D + mg = -kx - c\dot{x} + mg$$

$$\boxed{m\ddot{x} + c\dot{x} + kx = mg}$$

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = mg$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = mg$$

b. $\omega_n^2 = \frac{k}{m} \rightarrow \boxed{\omega_n = \sqrt{\frac{k}{m}}}$

c. $2\zeta\omega_n = \frac{c}{m} \rightarrow \boxed{\zeta = \frac{c}{2m\omega_n}}$

d. Assume $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = u$ (in full eq. of motion $u = mg$)

$$\left[s^2 + 2\zeta\omega_n s + \omega_n^2\right]\underline{X} = \underline{U}$$

$$\boxed{\frac{\underline{X}}{\underline{U}} = \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}}$$

## Problem 1 (cont.)

e. Found $X = \frac{1}{s^2+2\zeta\omega_n s+\omega_n^2} U$

A step in gravitational force means $U = \frac{mg}{s}$

$$X = \frac{mg}{s(s^2+2\zeta\omega_n s+\omega_n^2)}$$

To find $x(t)$ from this, we need to take the inverse Laplace Transform

The closest form in the inverse Laplace Table is:

$$X = \frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}$$

This is just $\frac{\omega_n^2}{mg}\left(\frac{mg}{s(s^2+2\zeta\omega_n s+\omega_n^2}\right)$

We know that multiplying by constants in the Laplace Domain is just multiplying by constants in the time domain, so

$$x(t) = \frac{mg}{\omega_n^2}\left(\mathcal{L}^{-1}\left[\frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}\right]\right)$$

$$\boxed{x(t) = \frac{mg}{\omega_n^2}\left(1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin(\omega_d t+\phi)\right) \quad \text{where} \quad \omega_d = \omega_n\sqrt{1-\zeta^2}, \quad \phi = \cos^{-1}\zeta}$$

f. Use the final value Theorem to find the steady-state value

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s) = \lim_{s\to 0} \cancel{s}\left[\frac{mg}{\cancel{s}(s^2+2\zeta\omega_n s+\omega_n^2)}\right] = \frac{mg}{\omega_n^2} = \frac{mg}{\left(\frac{K}{m}\right)} = \frac{m^2 g}{K}$$

$$\boxed{x_{ss} = \frac{m^2 g}{K}}$$

_____ **Problem 2 – 25 Points**

The system in Figure 2 connected to ground through a spring of spring constant $k$. It is also connected to input $x_d$ via a damper of damping coefficient $c$. At time $t = 0$, a unit-displacement step input is applied to $x_d$.

a. Write the equations of motion for this system.

b. Write the transfer function from the input $x_d$ to the position of the mass $x$. If you were unable to find the equation of motion in part a., assume it is $\ddot{x} + \omega_n^2 x = 2\zeta\omega_n(\dot{x}_d - \dot{x})$.

c. What is the steady-state value of $x(t)$?

d. What is the steady-state value of the tracking error, $x_d(t) - x(t)$?

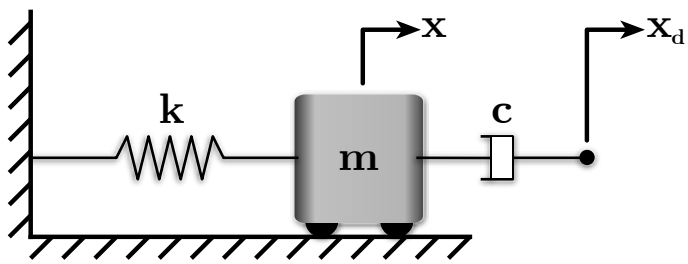
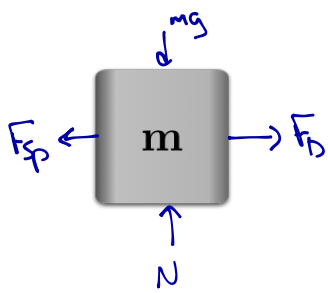



**Figure 2: A Mass-spring System with Damper to Input**

a.




$$m\ddot{x} = F_D - F_{sp}$$

$$F_D = c(\dot{x}_d - \dot{x}) \leftarrow \text{if } \dot{x}_d > \dot{x}$$

the direction of the force matches the FBD

$$m\ddot{x} = -kx + c(\dot{x}_d - \dot{x}) \qquad F_{sp} = kx$$

$$\boxed{m\ddot{x} + c\dot{x} + kx = c\dot{x}_d}$$

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = \frac{c}{m}\dot{x}_d$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 2\zeta\omega_n\dot{x}_d \quad \text{where } 2\zeta\omega_n = \frac{c}{m}, \quad \omega_n^2 = \sqrt{\frac{k}{m}}$$

b. Using $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 2\zeta\omega_n\dot{x}_d$ form

$$\left[s^2 + 2\zeta\omega_n s + \omega_n^2\right]X = \left[2\zeta\omega_n s\right]X_d$$

$$\boxed{\frac{X}{X_d} = \frac{2\zeta\omega_n s}{s^2 + 2\zeta\omega_n s + \omega_n^2}}$$

## Problem 2 (cont.)

c. $X = \left[\frac{2\zeta\omega_n s}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right] X_d$ A unit step in $X_d(t)$ at $t=0 \rightarrow X_d(s) = \frac{1}{s}$

$$X = \frac{2\zeta\omega_n s}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

Use the Final Value Theorem to find the steady-state position of $x(t)$

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$$

$$\boxed{\lim_{s\to 0} sX(s) = \cancel{s}\left[\frac{2\zeta\omega_n s}{\cancel{s}(s^2 + 2\zeta\omega_n s + \omega_n^2)}\right] = 0}$$

$\leftarrow$ x returns to its equil. position as $t \to \infty$

d. We want to use the Final Value Theorem again to find $e(t)$ as $t\to\infty$.

Intuitively, we know that for the unit-step input, given $\lim_{t\to\infty} x(t) = 0$, the steady state error should be 1, since $x_d(\infty) = 1$

$$E = X_d - X = \left[1 - \frac{2\zeta\omega_n s}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right] X_d = \left[\frac{s^2 + \omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right] X_d$$

If $X_d(s) = \frac{1}{s}$, $E = \frac{s^2 + \omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$

$$\boxed{\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \cancel{s}\left[\frac{s^2 + \omega_n^2}{\cancel{s}(s^2 + 2\zeta\omega_n s + \omega_n^2)}\right] = \frac{\omega_n^2}{\omega_n^2} = 1}$$

As our intuition predicted

_____ **Problem 3 – 50 Points**

The block diagram in Figure 3 represents a position control system where:

$$G_p(s) = \frac{1}{s(Js+b)}$$

and $K$ is a constant.

a. What is the open-loop transfer function this system?

b. What is the time response, $y(t)$, of the open-loop system to an impulse input (*i.e.* $r(t) = \delta(t)$)?

c. What is the closed-loop transfer function this system?

d. What is the natural frequency of the closed-loop system?

e. What is the damping ratio of the closed-loop system?

f. For a fixed natural frequency, what affects would increasing the damping have on the system response? If possible, provide support for your analysis.

g. For a ramp input, $r(t) = At, \forall t \geq 0$:

   i. Defining tracking error as $r(t) - y(t)$, what is the steady-state error of the open-loop system to this input?

   ii. What is the steady-state error for the closed-loop system to this input?

   iii. For the closed-loop system, how should $K$ be chosen to limit this error?

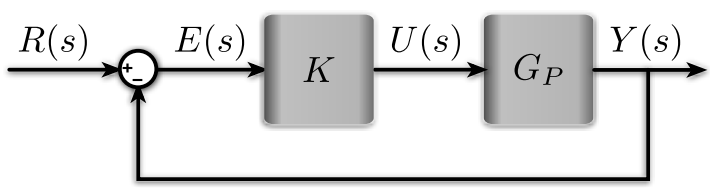
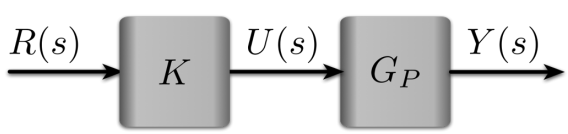



**Figure 3: Block Diagram of a Position Control System**

a. In the open-loop case there is no feedback, so the block diagram looks like:




So, the open-loop TF is just

$$\boxed{\frac{Y}{R} = \frac{K}{s(Js+b)}}$$

## Problem 3 (cont.)

b. To find the time response, write

$$Y = \frac{K}{s(Js+b)} R$$

For $r(t) = \delta(t)$, $R(s) = 1$, so

$$Y(s) = \frac{K}{s(Js+b)}$$ describes the response to the impulse input

To find $y(t)$ use the Inverse Laplace Transform

In this case, we need to use partial fraction expansion to match a form available in the lookup tables

$$Y(s) = \frac{a_1}{s} + \frac{a_2}{Js+b}$$

$$a_1 = \cancel{s}\left(\frac{K}{\cancel{s}(Js+b)}\right)\Bigg|_{s=0} = \frac{K}{b}$$

$$a_2 = \cancel{(Js+b)}\left(\frac{K}{s\cancel{(Js+b)}}\right)_{s=\frac{-b}{J}} = \frac{K}{\left(-\frac{b}{J}\right)} = \frac{-KJ}{b}$$

$$Y(s) = \frac{K/b}{s} + \frac{(-KJ/b)}{Js+b} = \frac{(K/b)}{s} - \frac{K/b}{s+b/J}$$

So $$\boxed{y(t) = \frac{K}{b}\left[1 - e^{-b/J\,t}\right]}$$

## Problem 3 (cont.)

c. For the closed-loop system

$$Y = \underbrace{\left[\frac{K}{s(Js+b)}\right]}_{\text{Call this } G} E \qquad \text{and} \qquad E = R - Y$$

$$Y = GE = G(R-Y)$$

$$(1+G)Y = GR \longrightarrow \frac{Y}{R} = \frac{G}{1+G}$$

We could have skipped directly here if we remembered this form

$$\boxed{\frac{Y}{R} = \frac{K}{s(Js+b)+K} = \frac{K}{Js^2+bs+K}}$$

d. Dividing by J in numerator and denominator

$$\frac{Y}{R} = \frac{K/J}{s^2 + \frac{b}{J}s + K/J} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

So, $\omega_n^2 = \frac{K}{J} \longrightarrow \boxed{\omega_n = \sqrt{\frac{K}{J}}}$

e. And, $2\zeta\omega_n = \frac{b}{J} \longrightarrow \boxed{\zeta = \frac{b}{2J\omega_n}}$

f. Increasing the damping ratio will
- reduce overshoot $\rightarrow PO = 100\, e^{-\zeta\pi/\sqrt{1-\zeta^2}} \rightarrow$ larger $\zeta \rightarrow$ smaller PO
- decrease settling time $\rightarrow T_s \approx \frac{4}{\zeta\omega_n} \rightarrow$ larger $\zeta \Rightarrow$ smaller $T_s$

g. $r(t) = t, \ \forall t > 0 \rightarrow R(s) = \frac{1}{s^2}$

i. $E = R - Y$ and $Y = \left[\frac{K}{s(Js+b)}\right] R \rightarrow E = \left[\frac{s(Js+b) - K}{s(Js+b)}\right] R =$

$$E = \left[\frac{s(Js+b) - K}{s(Js+b)}\right]\left[\frac{1}{s^2}\right]$$

Use the Final Value Theorem

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$$

$$\boxed{\lim_{s\to 0} sE(s) = \lim_{s\to 0} \cancel{s}\left[\frac{Js^2+bs-K}{(Js^2+bs)s^{\cancel{2}}}\right] = \infty}$$

## Problem 3 (cont.)

ii. For the closed-loop system, we found

$$\frac{Y}{R} = \frac{K}{Js^2+bs+K} \rightarrow Y = \frac{K}{Js^2+bs+K} R$$

$$E = R - Y = R - \left[\frac{K}{Js^2+bs+K}\right] R = \left[\frac{Js^2+bs}{Js^2+bs+K}\right] R = \left[\frac{Js^2+bs}{Js^2+bs+K}\right]\left[\frac{1}{s^2}\right]$$

To find the steady-state error, use the Final Value Theorem:

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$$

$$\boxed{\lim_{s\to 0} sE(s) = \cancel{s}\left[\frac{Js^2+bs}{\cancel{s^2}(Js^2+bs+K)}\right] = \frac{\cancel{s}(Js+b)}{\cancel{s}(Js^2+bs+K)} = \frac{b}{K}}$$

iii. To minimize error for the closed-loop system, K should be large so that $e(\infty) = \frac{b}{K}$ is small.

# Possibly Useful Equations

$$\bar{f} = m\bar{a}$$

$$I_0\bar{\alpha} = \sum \bar{M}_0$$

$$\sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b)$$

$$\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b)$$

$$e^{\pm i\omega t} = \cos(\omega t) \pm i\sin(\omega t)$$

$$x(t) = ae^{i\omega_n t} + be^{-i\omega_n t}$$

$$x(t) = a\cos\omega_n t + b\sin\omega_n t$$

$$x(t) = e^{-\zeta\omega_n t}\left[a\cos(\omega_d t) + b\sin(\omega_d t)\right]$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\int u\ dv = uv - \int v\ du$$

$$\delta_{oc}V = \forall \sum$$

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$$
$$\mathbf{y} = C\mathbf{x} + D\mathbf{u}$$

$$\mathbf{x} = \mathbf{\Phi}(t)\mathbf{x}(0) + \int_0^t \mathbf{\Phi}(t-\tau)\mathbf{B}\mathbf{u}(\tau)d\tau$$

$$\mathbf{X}(s) = [s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{x}(0) + [s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B}\mathbf{U}(s)$$

$$S_\alpha^T = S_G^T S_\alpha^G$$

$$T_s = \frac{4}{\zeta\omega_n}$$

$$T_r \approx \frac{2.16\zeta + 0.60}{\omega_n}, \quad 0.3 \leq \zeta \leq 0.8$$

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

$$M_{P_t} = 1 + e^{-\zeta\pi/\sqrt{1-\zeta^2}}$$

$$\%\text{Overshoot} = 100e^{-\zeta\pi/\sqrt{1-\zeta^2}}$$

$$V(\omega,\zeta) = e^{-\zeta\omega t_n}\sqrt{[C(\omega,\zeta)]^2 + [S(\omega,\zeta)]^2}$$

**Table 1: Laplace Transform Pairs**

| $f(t)$ | $F(s)$ |
|---|---|
| $f(t) = A,\ \forall t > 0$ | $\frac{A}{s}$ |
| $f(t) = At,\ \forall t > 0$ | $\frac{A}{s^2}$ |
| $\delta(t)$ | $1$ |
| $e^{-at}$ | $\frac{1}{s+a}$ |
| $\sin\omega t$ | $\frac{\omega}{s+\omega^2}$ |
| $\cos\omega t$ | $\frac{s}{s+\omega^2}$ |
| $e^{-at}\sin\omega t$ | $\frac{\omega}{(s+a)^2+\omega^2}$ |
| $e^{-at}\cos\omega t$ | $\frac{s+a}{(s+a)^2+\omega^2}$ |
| $\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}\right),\ \zeta < 1$ | $\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$ |
| $1 - \frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2} + \phi\right),\ \phi = \cos^{-1}\zeta,\ \zeta < 1$ | $\frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}$ |