# MCHE 474: Control Systems 

Fall 2017 - Final Exam
Monday, December 4

Name:


CLID: $\qquad$
Directions: Complete the attached problems making sure to clearly indicate your answer, show your work, and list any assumptions that you have made (with justification for them, if necessary). If you need extra space for any question, you may attach additional sheets of paper, which will be provided to you. No calculators are allowed.

## Academic Honesty (just a reminder):

An essential rule in every class of the University is that all work for which a student will receive a grade or credit be entirely his or her own or be properly documented to indicate sources. When a student does not follow this rule, $\mathrm{s} / \mathrm{he}$ is dishonest and $\mathrm{s} /$ he defeats the purpose of the course and undermines the goals of the University.
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## Problem 1-40 Points

The system in Figure 1 is a simple model representing a single-link robotic arm. There is a pure torque, $T$, applied at the shoulder joint. The angle of the arm is represented by angle $\theta$. Friction in the system is modeled as a rotational damper at the shoulder (The resulting torque is proportional to the angular velocity, $\left.T_{d}=c \dot{\theta}\right)$.
a. Write the equations of motion for this system.
b. Write the transfer function from the torque input $T$ to the position of the arm $\theta$. If you were unable to find the equation of motion in part a., assume it is $m l^{2} \ddot{\theta}+c \dot{\theta}=T$.
c. What is the time response, $\theta(t)$, of the system to an impulse torque input (ie. $T(t)=\delta(t))$ ?

In order to position the arm, you decide to use a Propertional Derivative (PD) feedback control system, using the desired arm angle, $\theta_{d}$, as the reference command, proportional gain $k_{p}$, and derivative gain $k_{d}$.


Figure 1: Simple Model of a Single-link Robotic Arm
d. Draw a block digram representing the proposed control system.
e. What is the closed-loop transfer function this system?
f. What is the natural frequency of the closed-loop system?
g. What is the damping ratio of the closed-loop system?
h. What is the steady-state error of the closed-loop system to a step input in desired angle (ie. $\left.\theta_{d}(t)=A, \forall t \geq 0\right)$ ?
i. Sketch the root locus for this system for the proportional gain, $k_{p}$. Assume that the ratio $\frac{k_{p}}{k_{d}}$ is constant and $\frac{k_{p}}{k_{d}}>\frac{c}{m l^{2}}$.
j. For the intended application, the desired system response is the fastest response possible that maintains near-zero overshoot. Indicate the position of the closed-loop poles that would result in such a response on the root locus you sketched.
a


$$
\begin{aligned}
& \sum \bar{\mu}_{0}: T-T_{d}=T-c \dot{\theta} \\
& {\left[I_{0} \ddot{\theta}=T-c \theta\right\} \bar{k}} \\
& I_{0}: M l^{2} \\
& M l^{2} \dot{\theta} \cdot c \dot{\theta}=T
\end{aligned}
$$

b. $\left(M l^{2} s^{2}+c s\right) \theta=T$


Problem 1 (cont.)
c. $\frac{\oplus}{T}=\frac{1}{M d^{2} s^{2}+c s}=\frac{1}{s\left(n p_{s}^{2}+c\right)} \quad$ a imply $\quad d(t)=1 \mathrm{~m}$ th Laplace domain

$$
\Theta(s)=\frac{1}{\left.s(m)^{2} s+c\right)} T(s)=\frac{1}{s\left(\mu D^{2} s+c\right)}
$$

Now, we need to use the mise laplace if to get $\theta(t)$
Well need to do partial fraction appassion

$$
\begin{aligned}
& \left(H(s)=\frac{a_{1}}{s}+\frac{a_{2}}{m l^{0} s+c}\right. \\
& a_{1}=\left.s\left(\frac{1}{s\left(m l^{2} s+c\right)}\right)\right|_{s=0}=\frac{1}{c} \\
& a_{2}=\left(m l^{2} s+c\right)\left(\left.\frac{1}{\left.s(m)^{2} s+c\right)}\right|_{s=\frac{c}{m l^{2}}}=\frac{-m l^{2}}{c}\right. \\
& \left(\#(s)=\frac{1 / c}{s}+\frac{-m l^{2} / c}{m l^{2} s+c}=\frac{1 / c}{s}-\frac{1 / c}{s+\frac{c}{m l^{2}}}\right. \\
& \theta(t)=\frac{1}{c}-\frac{1}{c} \exp \left[-\frac{c}{\left.m l^{2} t\right]}\right.
\end{aligned}
$$

Problem 1 (cont.)
d.

where

$$
\begin{aligned}
& R(s)=\Pi_{8}(s) \\
& U(s)=T(s) \\
& Y(s)=\Theta(s)
\end{aligned} \quad \text { and } \quad G_{c}(s): k_{p}+k_{d} s \text { and } \quad G_{p}(s)=\frac{1}{s\left(m^{0} s^{2}+c\right)}
$$

e.

$$
\begin{aligned}
& \text { e. } \frac{(\#)}{m d}: \frac{G_{0} C_{p}}{1+G C C_{p}}=\frac{k_{p}+k_{d} s}{m l^{2}+(c+k d) s+k_{p}}=\frac{1 / m \omega^{2}\left(k_{p}+k_{d s}\right)}{s^{2}+\left(\frac{c+k_{d} d}{m d^{2}}\right) s+\frac{k_{p}}{m l^{2}}} \\
& \text { f. Matching terms }
\end{aligned}
$$

$$
\omega_{n}^{2}=\frac{k_{p}}{m l^{2}} \rightarrow \omega_{n}=\sqrt{\frac{k_{p}}{m l^{2}}}
$$

g. $\quad 2 \Sigma_{\omega_{n}}=\frac{c+k_{d}}{m l^{2}} \rightarrow \Sigma: \frac{c+k_{d}}{2 m l^{2} w_{n}}$
h.

$$
\begin{aligned}
& E(s)=R-Y=\Theta_{d}-\Theta=\left[1-\frac{k_{p}+k_{d s}}{m D_{s}^{2}\left(c \cdot k_{d}\right) s^{2}+k_{p}}\right]_{d}=\left[\frac{m l^{2} s^{2}+c s}{m d^{2}+\left(c+k_{d}\right) s+k_{p}}\right] \oplus_{d} \\
& \theta_{d}(s)=\frac{A}{s} \text { for } \quad \Pi_{2}(t)=A \quad H>O \\
& \left.E(s)=\left\lfloor\frac{m b^{2} s^{2}+c s}{m d s^{2}+\left(c+k_{d}\right) s+k_{p}}\right] \backslash \frac{1}{s}\right\rceil \\
& \left.\lim _{t \rightarrow \infty} e(t)=\lim _{s \rightarrow 0} s E(s)=\lim _{s \rightarrow 0} s \left\lvert\, \frac{m l^{2} s^{2}+c s}{m l s^{2}+\left(c+k_{d}\right) s+k_{p}}\right.\right]\left|\frac{1}{\phi}\right|=0
\end{aligned}
$$

Problem 1 (cont.)
i. The open-kop $\pi$ is $\frac{k_{d s}+k_{p}}{s\left(m l_{s}^{2}+c\right)}=\frac{k_{p}\left(\frac{\left.k_{d} s+1\right)}{k_{p} s+1} \leftarrow 1 \alpha \text { zero at } \frac{-k_{p}}{k d}\right.}{s\left(m l^{s} s+c\right)} \longleftarrow 2 \alpha$ poles at 0 and $\frac{-c}{M D^{2}}$

From proton dexupten, $\frac{k_{p}}{k_{0}}, \frac{c}{\mathrm{ml}^{2}}$

- we shesild hal 2 spate lori

Asymptotes

$$
\begin{aligned}
& \sigma_{A}=\frac{-\frac{k P}{c a} \cdot \frac{c}{m D^{2}}}{2-1} \leftarrow \text { so somewhere in heaven } \\
& \text { Lelmots polo and the zero } \\
& \Phi_{A}=\left(\frac{\partial k+1}{n-1}\right) 180^{\circ}, k=0, n-n-1 \\
& =\left(\frac{2 k+1}{1}\right) 180^{\circ}, k=0 \rightarrow 180^{\circ}
\end{aligned}
$$

Braleawry Pouts

$$
\begin{aligned}
& k_{p}=\frac{\mathrm{m}^{3} s^{2}+c s}{T_{d s}+1} \quad \text { were } t_{d}=\frac{k_{c l}}{k_{p}} \equiv \text { contact } \\
& \frac{\partial k_{p}}{\partial s}=0
\end{aligned}
$$

$$
\begin{aligned}
& \rightarrow s=\frac{-c}{\partial n l^{2}}
\end{aligned}
$$

j. The closed-loop poles should be placed close to the real axis on the leftmost branch of the locus, indicated by the orange triangles on the locus above. At this point, the poles are approaching critically damped, so overshoot will be minimal. Increasing the gains further will move the poles to the real axis and will result in a slower, overdamped system response.
$\qquad$

## Problem 2-20 Points

Consider a system represented by the block diagram in Figure 2, where:

$$
G(s)=\frac{\omega_{n}^{2}}{s\left(s+2 \zeta \omega_{n}\right)}
$$

a. Write the closed-loop transfer function for this system.
b. What is the steady-state error of the response to a unit ramp input, $r(t)=t, \forall t>0$ ?

Now, $G_{f}(s)$ is added to filter the reference command, as shown in Figure 3. The filter is selected such that:

$$
G_{f}(s)=1+K s, \quad \text { where } K>0
$$

c. Write the closed-loop transfer function for this system.
d. What is the steady-state error of the response to a unit ramp input, $r(t)=t, \forall t>0$ ?
e. Can $K$ be selected to result in zero steady-state error? If so, what should it be?


Figure 2: A Unity Feedback System


Figure 3: A Unity Feedback System with Input Filter
a. $\frac{4}{R}=\frac{G}{1+G}=\frac{\omega_{n}^{2}}{s^{2}+2 \operatorname{con}_{n} S+\omega_{n}^{2}}$
b. $E: R-Y=R-G E \rightarrow(1+G) E=R \rightarrow E=\frac{1}{1+G} R$

$$
\begin{aligned}
& E=\frac{s\left(s+\partial\left(\omega_{n}\right)\right.}{s^{\partial} \partial\left(\omega_{n} s+\omega_{n}\right.} 2 R \quad R(s)=\frac{1}{s^{\partial}} \text { if } r(t)=t \theta t>0 \\
& E=\left[\frac{s\left(s+\partial\left[\omega_{n}\right)\right.}{s^{\partial} \partial\left(\omega_{n} s+\omega_{n}\right.}{ }^{2}\right]\left[\frac{1}{s^{\partial}}\right]
\end{aligned}
$$

Problem 2 (cont.)
b. (cont) $E=\left[\frac{s\left(s+\partial \sum \omega_{n}\right)}{s^{\partial} \partial\left(\omega_{n} s+\omega_{n}^{2}\right.}\right]\left\lfloor\frac{1}{s^{2}}\right]$

$$
\begin{aligned}
\lim _{t \rightarrow \infty} e(t)=\lim _{s \rightarrow 0} s E(s) & =\lim _{s \rightarrow 0} \delta\left[\frac{\delta\left(s+\partial\left\{\omega_{n}\right)\right.}{s^{\partial} \partial\left\{\omega_{n} s+\omega_{n}\right.}\right]\left[\frac{1}{s^{\partial}}\right] \\
& =\lim _{s \rightarrow 0} \frac{s+\partial \Sigma \omega_{n}}{s^{2} \partial\left\{\omega_{n} s+\omega_{n} \partial\right.}=\frac{\partial \Sigma}{\omega_{n}}
\end{aligned}
$$

c. $\bar{R}=\left(1+K_{S}\right) R$

$$
\begin{aligned}
& Y=G(\bar{R}-4)=G\left(\left(1+k_{s}\right) R-4\right) \\
& (1+G) Y=\left(1+K_{s}\right) G R \\
& \frac{4}{R}=\frac{\left(1+K_{s}\right) G}{1+G}=\frac{\left(1+K_{s}\right) \omega_{n}^{2}}{s^{2}+2 \delta \omega_{n} s+\omega_{n}^{2}}
\end{aligned}
$$

d. In this case the tracking error is nut just Es). We need to look at

$$
\begin{aligned}
& R-Y, E(s)=\bar{R}-Y . \\
& \bar{E}=R-Y=\left[1-\frac{(1+K s) \omega_{n}^{2}}{s^{2}+\partial \sum \omega_{n} s+\omega_{n}^{2}}\right] R=\left[\frac{s^{2}+\partial\left\{\omega_{n} s-K \omega_{n}^{2} s\right.}{s^{2}+\partial \omega_{n} s+\omega_{n}^{2}}\right] R \quad R=\frac{1}{s^{2}} \text { os hetore } \\
& \lim _{t \rightarrow 0} \bar{e}(t)=\lim _{s \rightarrow 0} s E(s)=\lim _{s \rightarrow 0} \$\left[\frac{s\left(s+\partial\left(\omega_{n}-K \omega_{n}^{2}\right)\right.}{s^{2}+\partial \omega_{n} s+\omega_{n}^{2}}\right]\left[\frac{1}{s^{2}}\right]=\frac{\partial\left\langle\omega_{n}-K \omega_{n}^{2}\right.}{\omega_{n}^{2}}
\end{aligned}
$$

e. $e_{s s}=\frac{\partial \sum \omega_{n}-K \omega_{n}{ }^{2}}{\omega_{n}{ }^{2}} \leftarrow$ we need the de, to be zero, so $\ldots$

$$
\partial \Sigma \omega_{n}=K \omega_{n}^{2} \rightarrow K=\frac{\Sigma}{\omega_{n}} \text { would rake } e_{s s}=0
$$

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## Problem 3-20 Points

Consider a system represented by the block diagram in Figure 4, where:

$$
G(s)=\frac{s+3}{s(s+1)} \quad \text { and } \quad G_{c}(s)=K(s+2)
$$

a. Write the open-loop transfer function for this system.
b. Write the closed-loop transfer function for this system.
c. Sketch the root locus of the system.
d. Is the system unstable for any value of $K$ ? Support your answer using your root locus sketch.
e. On the root locus you sketched, indicate the regions that correspond to an overdamped response.
f. On the root locus you sketched, indicate the regions that correspond to an underdamped response.


Figure 4: Feedback Control System Block Diagram
a. $O C=G_{c} O_{p}=\frac{K(s+2)(s+3)}{s(s+1)}$
b. $C L=\frac{G_{c} G p}{1+G_{c} O p}=\frac{K(s+2)(s+3)}{s(s+1)+K(s+2)(s+3)}$

Problem 3 (cont.)
c. $\alpha=G_{C} G P=\frac{k(s+2)(s+3)}{s(s+1)} \longleftarrow$ 2000 at -2 and -3

\# peles = \# zeas, so no cruppotes

Brakconcy Ponts

$$
\begin{aligned}
& 1+\frac{K(s+2)(s+3)}{5(s+1)}=0 \\
& K=-\frac{s(s+1)}{(s+2)(s+3)}=-\frac{3^{2}+s}{(s+2)(s+3)} \\
& \frac{\partial K}{\partial s}=0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial k}{\partial s} & =-\frac{(2 s+1)}{(s+2)(s+3)}+\frac{\left(-s^{2}+s\right)}{(s+2)^{2}(s+3)}+\frac{\left(-s^{2}+s\right)}{(s+2)(s+3)^{2}}=0 \\
& =-\frac{(2 s+1)(s+2)(s+3)+(-s(s+1)(s+3))}{(s+2)^{2}(s+3)^{2}}+(-s(s+1)(s+2)) \\
& =\frac{-(2 s+1)(s+2)(s+3)+(-s(s+1))(2 s+5)}{(s+2)^{2}(s+3)^{2}}=0 \\
& =\frac{-\left(2 s^{2}+4 s+s+2\right)(s+3)+\left(-s\left(2 s^{2}+5 s+2 s+5\right)\right)}{(s+2)^{2}(s+3)^{2}}=-\frac{\left(2 s^{2}+5 s+2\right)(s+3)-2 s^{3}-7 s^{2}-5 s}{(s+2)^{2}(s+3)^{2}} \\
& =-\frac{\left.\left(2 s^{3}+5 s^{2}+2 s+6 s^{2}+15 s+6\right)-2 s^{2}-7 s^{2}-5 s\right)}{(s+2)^{2}(s+3)^{2}}=\frac{4 s^{2}+12 s+6}{(s+2)^{2}(s+3)^{2}}=0
\end{aligned}
$$

Nole: You calld jut set up $\frac{\partial k}{\partial s}=0$ and oppore poonts knowim that they hor to occor hetween poles ard
find $s=-0.63$ own $s=-2.37$ ore posille points between reros.

Problem 3 (cont.)

d. This system is stable for any de of $K, 0 \leqslant K \leqslant \infty$. As sean $m$ the rot $k c u)$, the closeetoop poles never enter the right -hard play
e. The regis of the locks an the seal axis carrepand to cuerdomped reppanse. They ore showed in green drove.
f. The region of the locus with complex-conjugate pars as roots correspond to undercomped repanal. They ar highlighted in red above.
$\qquad$

## Bonus - 10 Points

As part of a consulting project, you've been given the Bode Plot shown in Figure 5 and told the damping ratio is $\zeta=0.01$ and asked to design a vibration-reducing controller for the system.
a. What is the approximate natural frequency of this system?
b. Roughly sketch the response of this system to a step input.
c. The frequency is well known, so you've decided to use a ZV Input Shaper. Write the form of the ZV shaper for this system.
d. Roughly sketch the resulting ZV-shaped step input and the response to it.


Figure 5: Bode Plot for a Lightly-damped Flexible System
a. The peak e in the regnituct plot and phone chang in the phone plot happen at = $1 \mathrm{~Hz}_{2}$. That is the nature freq. $\omega_{n} \hat{=} 1 \mathrm{H}_{2}$

Bonus (cont.)
b. The sypten shoulo har a lightly-comped, osillatom reganel to a step mput

c.

$$
\begin{aligned}
& Z V=\left[\begin{array}{l}
A_{i} \\
A_{i}
\end{array}\right]=\left[\begin{array}{cc}
\frac{1}{1+k} & \frac{k}{1+k} \\
0 & \tau_{d} / \partial
\end{array}\right] \quad \begin{array}{l}
\left.K=\exp \mid \sqrt{\sqrt[-\pi]{1-\varepsilon^{2}}}\right] \\
\tau_{d} \hat{\wedge} / s \quad \tau_{d} / \partial=0.5 s
\end{array} \\
& Z V=\left|\begin{array}{l}
A_{i} \\
t_{i}
\end{array}\right| \approx\left[\begin{array}{cc}
1 / 2 & 1 / 2 \\
0 & 1 / 2
\end{array}\right] \quad \text { For } \sum=0.01 \quad K \approx 1
\end{aligned}
$$

d.


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## Possibly Useful Equations

$$
\begin{array}{c|c}
\bar{f}=m \bar{a} & \dot{\mathbf{x}}=A \mathbf{x}+B \mathbf{u} \\
\mathbf{y}=C \mathbf{x}+D \mathbf{u} \\
I_{0} \bar{\alpha}=\sum \bar{M}_{0} \\
\sin (a \pm b)=\sin (a) \cos (b) \pm \cos (a) \sin (b) & \mathbf{x}=\mathbf{\Phi}(t) \mathbf{x}(0)+\int_{0}^{t} \mathbf{\Phi}(t-\tau) \mathbf{B u}(\tau) d \tau \\
\cos (a \pm b)=\cos (a) \cos (b) \mp \sin (a) \sin (b) & \mathbf{X}(s)=[s \mathbf{I}-\mathbf{A}]^{-1} \mathbf{x}(0)+[s \mathbf{I}-\mathbf{A}]^{-1} \mathbf{B U}(s) \\
e^{ \pm i \omega t}=\cos (\omega t) \pm i \sin (\omega t) & S_{\alpha}^{T}=S_{G}^{T} S_{\alpha}^{G} \\
x(t)=a e^{i \omega_{n} t}+b e^{-i \omega_{n} t} & T_{s}=\frac{4}{\zeta \omega_{n}} \\
x(t)=a \cos \omega_{n} t+b \sin \omega_{n} t \\
x(t)=e^{-\zeta \omega_{n} t}\left[a \cos \left(\omega_{d} t\right)+b \sin \left(\omega_{d} t\right)\right] \\
x=\frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a} & T_{r} \approx \frac{2.16 \zeta+0.60}{\omega_{n}}, \quad 0.3 \leq \zeta \leq 0.8 \\
\int u d v=u v-\int v d u \\
T_{p}=\frac{\pi}{\omega_{n} \sqrt{1-\zeta^{2}}} \\
\delta_{o c} V=\forall \sum & M_{P_{t}}=1+e^{-\zeta \pi / \sqrt{1-\zeta^{2}}} \\
\% \mathrm{Overshoot}=100 e^{-\zeta \pi / \sqrt{1-\zeta^{2}}} \\
V(\omega, \zeta)=e^{-\zeta \omega t_{n}} \sqrt{[C(\omega, \zeta)]^{2}+[S(\omega, \zeta)]^{2}}
\end{array}
$$

Table 1: Laplace Transform Pairs

| $f(t)$ | $F(s)$ |
| :---: | :---: |
| $f(t)=A, \forall t>0$ | $\frac{A}{s}$ |
| $f(t)=A t, \forall t>0$ | $\frac{A}{s^{2}}$ |
| $\delta(t)$ | 1 |
| $e^{-a t}$ | $\frac{1}{s+a}$ |
| $\sin \omega t$ | $\frac{\omega}{s+\omega^{2}}$ |
| $\cos \omega t$ | $\frac{s}{s+\omega^{2}}$ |
| $e^{-a t} \sin \omega t$ | $\frac{\omega}{(s+a)^{2}+\omega^{2}}$ |
| $e^{-a t} \cos \omega t$ | $\frac{s+a}{(s+a)^{2}+\omega^{2}}$ |
| $\frac{\omega_{n}^{2}}{s^{2}+2 \zeta \omega_{n} s+\omega_{n}^{2}}$ |  |
| $1-\frac{1}{\sqrt{1-\zeta^{2}}} e^{-\zeta \omega_{n} t} \sin \left(\omega_{n} \sqrt{1-\zeta^{2}}+\phi\right), \phi=\cos ^{-1} \zeta, \zeta<1$ | $\frac{\omega_{n}^{2}}{s\left(s^{2}+2 \zeta \omega_{n} s+\omega_{n}^{2}\right)}$ |

## Possibly Useful Equations

$$
\begin{gathered}
{\left[\begin{array}{c}
A_{i} \\
t_{i}
\end{array}\right]=\left[\begin{array}{cc}
\frac{1}{1+K} & \frac{K}{1+K} \\
0 & \frac{\pi}{\omega \sqrt{1-\zeta^{2}}}
\end{array}\right]} \\
{\left[\begin{array}{c}
A_{i} \\
t_{i}
\end{array}\right]=\left[\begin{array}{ccc}
\frac{1}{1+2 K+K^{2}} & \frac{2 K}{1+2 K+K^{2}} & \frac{K^{2}}{1+2 K+K^{2}} \\
0 & \frac{\pi}{\omega \sqrt{1-\zeta^{2}}} & \frac{2 \pi}{\omega \sqrt{1-\zeta^{2}}}
\end{array}\right]} \\
K=e^{\frac{-\zeta \pi}{\sqrt{1-\zeta^{2}}}}
\end{gathered}
$$

